Lecture 24: Review for 2nd Exam

Math 3013
Oklahoma State University

March 25, 2022

Lecture 25: Review Session for Exam 2

Agenda:

1. Overiew of the Topics to be Covered on the Exam
2. Practice Exam

I. Subspaces, Bases, and Linear Independence

A. **Def.** A **subspace** is a set of vectors W that is closed under both scalar multiplication and vector addition:

   (i) $\lambda \in \mathbb{R}\ ,\ \mathbf{w} \in W \quad \Rightarrow \quad (\lambda \mathbf{w}) \in W$

   (ii) $\mathbf{w}_1, \mathbf{w}_2 \in W \quad \Rightarrow \quad (\mathbf{w}_1 + \mathbf{w}_2) \in W$

B. **Def.** A **basis** for a subspace W is a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ such that

   (i) every vector $\mathbf{w} \in W$ can be expressed as

$$\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \qquad (*)$$

   (ii) The coefficients $c_1, \ldots, c_k$ in (*) are unique.

C. **Def.** A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$are **linearly independent** if the only solution of

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

is $x_1 = 0, \ldots, x_k = 0$.

Subspaces, Bases, and Linear Independence, Cont'd

D. **Theorem:** A set of vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is a basis for a subspace W if and only if
   (i) $W = span(\mathbf{w}_1, \ldots, \mathbf{w}_k)$
   (ii) The vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are linearly independent.

E. **Theorem:** Every subspace has a basis and every basis for a subspace has the same number of vectors

F. **Def.** The **dimension** of a subspace W is the number of vectors in any basis for W.

Subspaces attached to a matrix and their bases

Suppose $\mathbf{A}$ is an $n \times m$ matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} \leftarrow & \mathbf{R}_1 & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & \mathbf{R}_n & \rightarrow \end{bmatrix} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{C}_1 & \cdots & \mathbf{C}_m \\ \downarrow & & \downarrow \end{bmatrix}$$

A. The **row space** of an $n \times m$ matrix $\mathbf{A}$ is the subspace

$$RowSp\,(\mathbf{A}) = \{t_1\mathbf{R}_1 + \cdots + t_n\mathbf{R}_n \mid t_1, \ldots, t_n \in \mathbb{R}\} \subset \mathbb{R}^m$$

A basis for $RowSp(\mathbf{A})$ is found by row reducing $\mathbf{A}$ to R.E.F. and grabbing the non-zero rows of the R.E.F.

B. The **column space** of an $n \times m$ matrix $\mathbf{A}$ is the subspace

$$ColSp\,(\mathbf{A}) == \{t_1\mathbf{C}_1 + \cdots + t_m\mathbf{C}_m \mid t_1, \ldots, t_m \in \mathbb{R}\} \subset \mathbb{R}^n$$

A basis for $ColSp(\mathbf{A})$ is found by row reducing $\mathbf{A}$ to a R.E.F. $\mathbf{A}'$ and then grabbing the columns of $\mathbf{A}$ that correspond to columns of $\mathbf{A}'$ that contain pivots.

Subspaces attached to a matrix and their bases, Cont's

C. **Def.** The **null space** of an $n \times m$ matrix $\mathbf{A}$ is the solution set of the linear system $\mathbf{Ax} = \mathbf{0}$

$$NullSp(\mathbf{A}) \equiv \{\mathbf{x} \in \mathbb{R}^m \mid \mathbf{Ax} = \mathbf{0}\}$$

A basis for $NullSp(\mathbf{A})$ is found by expanding the solution of $\mathbf{Ax} = \mathbf{0}$ in terms of its free parameters and grabbing the constant vectors in that expansion.

D. **Def.** The **rank** of an $n \times m$ matrix $\mathbf{A}$ is the common dimension of its row and column spaces.

$$Rank(\mathbf{A}) \equiv dim(RowSp(\mathbf{A})) = dim(ColSp(\mathbf{A}))$$

II. Linear Transformations

A. **Def.** A **linear transformation** is a function $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

  (i) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.

  (ii) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$

B. **Theorem:** Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation and $\mathbf{A}_T$ is the $n \times m$ matrix formed by using the vectors $T([1, 0, \ldots, 0]), \ldots, T([0, \ldots, 0, 1]) \in \mathbb{R}^n$ as columns. Then

$$T(\mathbf{x}) = \mathbf{A}_T \mathbf{x} \qquad \forall \mathbf{x} \in \mathbb{R}^m$$

Linear Transformations, Cont'd

C. **Def.** The **range** of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is the following subspace of the codomain $\mathbb{R}^n$

$$Range\,(T) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T\,(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\}$$

One finds a basis for $Range(T)$ by finding a basis for the column space of $\mathbf{A}_T$.

D. **Def.** The **kernel** of a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is the following subspace of the domain $\mathbb{R}^m$

$$Ker\,(T) = \{\mathbf{x} \in \mathbb{R}^m \mid T\,(\mathbf{x}) = \mathbf{0}_{\mathbb{R}^n}\}$$

One finds a basis for $Ker(T)$ by finding a basis for the null space of $\mathbf{A}_T$.