Lecture 25 : Determinants

Math 3013
Oklahoma State University

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Agenda:

1. Fundamental Theorem of Invertible Matrices
2. Determinants of $n \times n$ Matrices; $n \leq 3$
3. Recursive Formula for Determinants
4. Determinants as Sums Over Permutations
5. Properties of Determinants

The Fundamental Theorem of Invertible Matrices

Recall the following theorem which shows the many connections between various problems we have considered.

Theorem

Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are equivalent (i.e., if any one of these statements is true, then all are true).

(a) $\mathbf{A}^{-1}$ *exists.*

(b) *Every linear system* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

(c) $\mathbf{Ax} = \mathbf{0}$ *has only the trivial solution* $\mathbf{x} = \mathbf{0}$*.*

(d) *The R.R.E.F. of* $\mathbf{A}$ *is the* $n \times n$ *identity matrix.*

(e) *The rank of* $\mathbf{A}$ *is* n*.*

Today we'll add a new item to this list of equivalent situations.

(f) $\det(\mathbf{A}) \neq 0$

The Determinant Function for Small Matrices

The proof of the original theorem was based on connections arising from the common row reduction algorithm by which the objects in (a) – (e) can be computed.

The new statement (f) will based instead on a particular polynomial function of the entries of a matrix $\mathbf{A}$, the so-called **determinant** function

$$\det : \{ n \times n \text{ matrices } \} \longrightarrow \mathbb{R}$$

In the next set of slides, I'll show how, for small n, one can define a function that tests for invertibility.

We'll then generalize the construction of the determinant function for arbitary square matrices.

The Determinant of a 1×1 matrix

Consider a 1×1 matrix $\mathbf{A} = [a]$
Such a matrix is invertible if and only if $a \neq 0$;

$$\mathbf{AB} = \mathbf{I} \iff [a][b] = [ab] = [1] \iff b = 1/a$$

Thus, if we thus define

$$det([a]) \equiv a$$

then we have

$$det([a]) \neq 0 \quad \Longrightarrow \quad [a] \text{ is invertible}$$

as our first example of statement (f) in the revised theorem.

The Determinant of a 2×2 Matrix

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

From the Fundamental Theorem

$$\mathbf{A}^{-1} \text{ exists} \quad \Leftrightarrow \quad R.R.E.F.(\mathbf{A}) = \mathbf{I}$$

Let's see what this requires of the entries a, b, c, d :

The Determinant of a 2×2 Matrix, Cont'd

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{a}R_1} \begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{R_2 \to R_2 - cR_1} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{cb}{a} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{cb}{a} \end{bmatrix} \xrightarrow{R_2 \to aR_2} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & ad - bc \end{bmatrix}$$

The next step would be to multiply the second row by $\frac{1}{ad-bc}$ so that its pivot becomes 1. However, we can't do that if $ad - bc = 0$ (we can't divide by 0). Thus,

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ can be row reduced to } \mathbf{I} \text{ only if } ad - bc \neq 0$$

The Determinant of a 2×2 Matrix, Cont'd

Thus, if

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \equiv ad - bc$$

then

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \neq 0 \quad \Rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \text{ exists}$$

and we have another example of statement (f)

The Determinant of a 3×3 Matrix

Now consider

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

A similar, but much longer, row reducibility argument shows that if

$$\det(\mathbf{A}) \equiv a(ei - fh) - b(di - fg) + c(dh - eg)$$

then

$$\det(\mathbf{A}) \neq 0 \quad \Rightarrow \quad \mathbf{A}^{-1} \text{ exists}$$

Determinants of Large Matrices

For larger matrices, similar row reduction computations become too unwieldy to identify corresponding determinant functions.

So instead we'll try to identify a pattern we find in the cases $n = 1, 2, 3.$, and then generalize that pattern.

Matrix Minors and Determinants

Definition

Let $\mathbf{A}$ be an $n \times n$ matrix. The $(ij)^{th}$-**minor** of $\mathbf{A}$ is the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and j^{th} column of $\mathbf{A}$.

Let us use the notation $\mathbf{M}_{ij}(\mathbf{A})$ to indicate the $(ij)^{th}$-minor of a matrix $\mathbf{A}$. E.g., if

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\mathbf{M}_{11} = \begin{bmatrix} e & f \\ h & i \end{bmatrix} \quad , \quad \mathbf{M}_{12} = \begin{bmatrix} d & f \\ g & i \end{bmatrix} \quad , \quad \mathbf{M}_{13} = \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Since $\mathbf{A}$ is 3×3

$$\begin{aligned} \det(\mathbf{A}) &\equiv a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= a \det(\mathbf{M}_{11}) - b \det(\mathbf{M}_{12}) + c \det(\mathbf{M}_{13}) \end{aligned}$$

Thus, we can express $\det(\mathbf{A})$ as a certain linear combination of the determinants of its minors :

This also works for 2×2 matrices

$$\begin{aligned} \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &\equiv ad - bc \\ &= a \det([d]) - b \det([c]) \\ &= a \det(\mathbf{M}_{11}) - b \det(\mathbf{M}_{12}) \end{aligned}$$

This generalizes as follows

Definition

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

be an $n \times n$ matrix and let $\mathbf{M}_{ij} = \mathbf{M}_{ij}(\mathbf{A})$, $i, j = 1, \ldots n$, denote its $(ij)^{th}$-minors. Then

$$\det(\mathbf{A}) \equiv a_{11} \det(\mathbf{M}_{11}) - a_{12} \det(\mathbf{M}_{12}) + \cdots + (-1)^{1+n} a_{1n} \det(\mathbf{M}_{1n})$$

Example

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

Compute $\det(\mathbf{A})$.

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}\det(\mathbf{M}_{11}) - a_{12}\det(\mathbf{M}_{12}) \\ &\quad + a_{13}\det(\mathbf{M}_{13}) - a_{14}\det(\mathbf{M}_{14}) \\ &= (1)\det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} - (0)\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \\ &\quad + (2)\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - (0)\det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \end{aligned}$$

The Cofactor Expansion Formulas for $det(\mathbf{A})$

We'll complete this calculation in a minute, but first let me give an even more general rule.

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *matrix and let* $\mathbf{M}_{ij}(\mathbf{A})$, $i, j = 1, \ldots n$, *denote its* $(ij)^{th}$*-minors. Then*

$$\det(\mathbf{A}) \equiv \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \quad \text{for each } i = 1, \ldots, n$$

and/or

$$\det(\mathbf{A}) \equiv \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \quad \text{for each } j = 1, \ldots, n$$

Cofactors: Notation and Nomenclature

The numbers

$$\mathbf{C}(\mathbf{A})_{ij} \equiv (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \equiv \text{ the } (ij)^{th}\text{-cofactor of } \mathbf{A}$$

are called **cofactors** of **A**. Note that an $n \times n$ matrix has a total of n^2 cofactors (one for each ordered pair of indices i, j).

The first formula of the theorem

$$\det(\mathbf{A}) \equiv \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \quad \text{for each } i = 1, \ldots, n \quad (1)$$

is referred to as the **cofactor expansion of** $\det(\mathbf{A})$ along the i^{th} row. The second formula of the theorem

$$\det(\mathbf{A}) \equiv \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \quad \text{for each } j = 1, \ldots, n \quad (2)$$

is referred to as the **cofactor expansion of** $\det(\mathbf{A})$ along the j^{th} column.

Theorem $\Longrightarrow$ **No matter which row or column we use to compute** $\det(\mathbf{A})$**, we'll get the same result.**

Let's now use the more general cofactor expansions to complete the calculation of

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$$

So far we have, from a cofactor expansion along the first row,

$$\det(\mathbf{A}) = (1)\det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} - 0 + (2)\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - 0$$

Now

$$
\begin{aligned}
\det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} &= (0)\det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} - (0)\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\quad + (1)\det\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \\
&= 0 + 0 + (1)\left((1)(2) - (1)(0)\right) \\
&= 2
\end{aligned}
$$

where we have applied a cofactor expansion of the first row.

To compute

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

the simplest thing would be do a cofactor expansion along the second column (or second row) - because the first and last terms of the expansion will have 0 as a factor.

$$\begin{aligned} \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} &= 0 + (1)\,(-1)^{2+2} \det \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 0 \\ &= (1)\,(1)\,(1-1) \\ &= 0 \end{aligned}$$

Thus,

$$\begin{aligned}\det(\mathbf{A}) &= (1)\det\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} + (2)\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= (1)(2) + (2)(0) \\ &= 2\end{aligned}$$

Soon we'll develop more efficient ways of calculating determinants.

But before doing that let me give another formula for the determinant that displays some of its most notable properties as function manifest.

Digression: Permutations of n

Definition

A **permutation** of n is a listing of the numbers $1, 2, \ldots, n$ in a particular order. The set of all permutations of n will be denoted by S_n.

For example, the permutations of 3 are

$$S_3 \equiv \{[1,2,3]\ ,\ [1,3,2]\ ,\ [2,1,3]\ ,\ [2,3,1]\ ,\ [3,1,2]\ ,\ [3,2,1]\}$$

Note that the particular permutation $[1, 2, 3, \ldots, n]$ is just the **standard ordering** of the numbers 1 through n.

Definition

The **sign** (or parity) of a permutation $\sigma = [\sigma_1, \ldots, \sigma_n]$ is $(-1)^s$, where s is the number of pairs (σ_i, σ_j) where $i < j$ but $\sigma_i > \sigma_j$.

Example: Permutations of $[1, 2, 3]$

For $[1, 2, 3]$ the possible pairs are $\{1, 2\}, \{1, 3\}, \{2, 3\}$

$$
\begin{aligned}
sgn([1,2,3]) &= (-1)^{0+0+0} = 1 \\
sgn([1,3,2]) &= (-1)^{0+0+1} = -1 \\
sgn([2,1,3]) &= (-1)^{1+0+0} = -1 \\
sgn([2,3,1]) &= (-1)^{1+1+0} = 1 \\
sgn([3,1,2]) &= (-1)^{0+1+1} = 1 \\
sgn([3,2,1]) &= (-1)^{1+1+1} = -1
\end{aligned}
$$

Combinatorial Formula for $\det(\mathbf{A})$

Here is a formula for the determininant of an $n \times n$ matrix in terms of permutations of $[1, 2, \ldots, n]$

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *matrix with entries* $a_{i,j}$. *Then*

$$\det(\mathbf{A}) = \sum_{\substack{\text{permutations} \\ \sigma}} sgn(\sigma)\, a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}$$

Example, Cont'd

Consider

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

We have

$$S_3 \equiv \{[1,2,3],[1,3,2],[2,1,3],[2,3,1],[3,1,2],[3,2,1]\}$$

So

$$\begin{aligned} \det(\mathbf{A}) &= sgn([1,2,3])\, a_{11}a_{22}a_{23} + sgn([1,3,2])\, a_{11}a_{23}a_{32} \\ &\quad + sgn([2,1,3])\, a_{12}a_{21}a_{33} + sgn([2,3,1])\, a_{12}a_{23}a_{3,1} \\ &\quad + sgn([3,1,2])\, a_{13}a_{21}a_{32} + sgn([3,2,1])\, a_{13}a_{22}a_{31} \\ &= aei - afg - bdi + bfg + cdh - ceg \\ &= a(ei - fg) - b(di - fg) + c(dh - eg) \\ &= a\det(\mathbf{M}_{11}) - b\det(\mathbf{M}_{12}) + c\det(\mathbf{M}_{13}) \end{aligned}$$

Corollary (Properties of Determinants)

Let $\mathbf{A}$ *be an* $n \times n$ *matrix.*

- *The determinant of an* $n \times n$ *matrix* $\mathbf{A}$ *is a polynomial of degree* n *in the entries of* $\mathbf{A}$.
- *In general, this polynomial has* $n!$ *terms*
- *Each term of this polynomial has exactly* n *factors; with each factor coming from a distinct row and distinct column of* $\mathbf{A}$.

These statements all follow from the formula

$$\det(\mathbf{A}) = \sum_{\substack{\text{permutations} \\ \sigma}} sgn(\sigma)\, a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}$$

The Complexity of Determinant Functions

Note that the polynomial corresponding to the determinant of a 5×5 matrix is going to involve $5! = 120$ individual terms.

For a 6×6 matrix, there will be $6! = 720$ different terms.

So the computation of determinants gets very strenuous, very quickly even for relatively small matrices.

In the next lecture, we will develop another, usually more expedient, way of computing determinants.