Lecture 26 : Determinants via Row Reduction

Math 3013
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Agenda:

1. Determinants: Results from the preceeding lecture
2. Determinants of Upper Triangular Matrices
3. Computing Determinants Using Row Reduction

Determinants

The **determinant** of an $n \times n$ matrix is a polynomial function of the entries of $\mathbf{A}$ with the property that

$$\det(\mathbf{A}) = 0 \quad \Longleftrightarrow \quad \begin{cases} \mathbf{A}^{-1} \text{ does not exist} \\ \mathbf{Ax} = \mathbf{b} \text{ can have more than one solution} \\ \mathbf{Ax} = \mathbf{0} \text{ has solutions other than } \mathbf{x} = \mathbf{0} \\ \mathbf{A} \text{ is } \textbf{not} \text{ row reducible to the identity matrix} \\ \mathit{Rank}(\mathbf{A}) < n \end{cases}$$

Determinant Formulas for Small Matrices ($n \leq 2$)

$\mathbf{n = 1}$:

$$\det([a_{11}]) = a_{11}$$

$\mathbf{n = 2}$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}$$

Recursive Formula for Larger Matrices

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *matrix.*

(i) *For any row index* i,

$$\det(\mathbf{A}) \equiv \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \quad \text{for each } i = 1, \ldots, n$$

(the cofactor expansion of $\det(\mathbf{A})$ *along the* i^{th} *row)*

(ii) *For any column index* j

$$\det(\mathbf{A}) \equiv \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \quad \text{for each } j = 1, \ldots, n$$

(the cofactor expansion of $\det(\mathbf{A})$ *along the* j^{th} *column)*

where $\mathbf{M}_{ij}$ *is the ij-**minor** of* $\mathbf{A}$ *(the* $(n-1) \times (n-1)$ *matrix formed by deleting the* i^{th} *row and* j^{th} *column of* $\mathbf{A}$*.*

Example: Determinant of a 3×3 Matrix

Thus, for example,

$$\begin{aligned}
\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) &= a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) \\
&\quad -a_{12}\det\left(\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}\right) \\
&\quad +a_{13}\det\left(\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\right) \\
&= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) \\
&\quad -a_{21}\left(a_{21}a_{32} - a_{23}a_{31}\right) \\
&\quad +a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)
\end{aligned}$$

Combinatorial Formula for det (A)

Here is a formula for the determininant of an $n \times n$ matrix in terms of permutations $[\sigma_1, \ldots, \sigma_n]$ of $[1, 2, \ldots, n]$

Theorem

Let **A** *be an* $n \times n$ *matrix with entries* $a_{i,j}$. *Then*

$$\det(\mathbf{A}) = \sum_{\substack{\textit{permutations} \\ \sigma}} sgn(\sigma)\, a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}$$

where, for a given permutation $\sigma = [\sigma_1, \ldots, \sigma_n]$

$$sgn(\sigma) = (-1)^{\#\text{times } \iota < j \text{ but } \sigma_i > \sigma_j}$$

Properties of Determinants

The formula

$$\det(\mathbf{A}) = \sum_{\substack{permutations \\ \sigma}} sgn(\sigma)\, a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}$$

shows that

- The determinant of an $n \times n$ matrix $\mathbf{A}$ is a polynomial of degree n in the entries of $\mathbf{A}$.
- Each term of this polynomial has exactly n factors; with each factor coming from a distinct row and distinct column of $\mathbf{A}$.
- In general, this polynomial has $n!$ terms

Determinants of Upper Triangular Matrices

We'll now develop some more efficient ways of computing determinants.
We'll begin with the special case of upper triangular matrices.

Definition

A $n \times n$ matrix $\mathbf{A}$ is called upper triangular if

$$j < i \quad \Rightarrow \quad (\mathbf{A})_{ij} = 0 \tag{*}$$

This condition forces every entry to the left of the diagonal entries a_{ii} to be zero. Thus, the non-zero entries of $\mathbf{A}$ must lie in the upper right hand corner of $\mathbf{A}$.

Example: An Upper Triangular Matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is upper triangular.

Note that being upper triangular is a slightly weaker condition than being in R.E.F..
As the example above shows, an upper triangular matrix is not necessarily a matrix in R.E.F.

On the other hand, a matrix in R.E.F. is always upper triangular.

Determinants of Upper Triangular Matrices, Cont'd

Theorem

Suppose $\mathbf{A}$ *is upper triangular. Then*

$$\det(\mathbf{A}) = a_{11}a_{22}\cdots a_{nn}$$

(i.e. $\det(\mathbf{A})$ *is just the product of the diagonal entries of* $\mathbf{A}$*).*

Idea of the Proof. Consider

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

Example: Determinant of an Upper Triangular Matrix

Carrying out a cofactor expansion along the last row we get (for the example above)

$$\begin{aligned}
\det(\mathbf{A}) &= 0\,(-1)^{4+1}\det(\mathbf{M}_{4,1}) + 0\,(-1)^{4+2}\det(\mathbf{M}_{4,2}) \\
&\quad +0\,(-1)^{4+3}\det(\mathbf{M}_{4,3}) + (a_{44})\,(-1)^{4+4}\det(\mathbf{M}_{4,4}) \\
&= 0+0+0+(a_{44})\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \\
&= (a_{44})\left((0)\det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix} - (0)\det\begin{pmatrix} a_{11} & a_{13} \\ 0 & a_{23} \end{pmatrix}\right. \\
&\quad \left. + (a_{33})\det\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\right) \\
&= (a_{44})\,(a_{33})\,(a_{11}\det([a_{22}]) - a_{12}\det([0])) \\
&= a_{44}a_{33}a_{22}a_{11}
\end{aligned}$$

However, the preceding simple formula for $\det(\mathbf{A})$ can only be applied when $\mathbf{A}$ is upper triangular.

OTHO, every matrix can be converted into an upper triangular matrix using row reduction (as R.E.F.'s are always upper triangular).

So how do elementary row operations affect determinants?

Elementary Row Operations and Determinants

Theorem: Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathcal{R}$ be an elementary row operation.

- If $\mathcal{R}$ is of the type $R_i \longleftrightarrow R_j$ (row interchange)

$$\det(\mathcal{R}(\mathbf{A})) = -\det(\mathbf{A})$$

- If $\mathcal{R}$ is of the type $R_i \to \lambda R_i$ (row rescaling)

$$\det(\mathcal{R}(\mathbf{A})) = \lambda \det(\mathbf{A})$$

- If $\mathcal{R}$ is of the type $R_i \to R_i + \lambda R_j$

$$\det(\mathcal{R}(\mathbf{A})) = \det(\mathbf{A})$$

So while elementary row operations do affect determinants, they only modify them by simple multiplicative factors.

Theorem

Suppose $\mathbf{A}$ *row reduces to a matrix* $\mathbf{A}'$ *in Row Echelon Form. Then*

$$\det(\mathbf{A}) = (-1)^r \frac{1}{\lambda_1 \cdots \lambda_k} \left(a'_{11} a'_{22} \cdots a'_{nn} \right)$$

where

- r *is the number of row interchanges used in the row reduction,*
- $\lambda_1, \ldots, \lambda_k$ *are the row rescaling factors used, and*
- $a'_{11}, \ldots, a'_{nn}$ *are the diagonal elements of the* $\mathbf{A}' = R.E.F.(\mathbf{A})$.

Example

Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

We have

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 3 & 0 & 1 \end{bmatrix} \xrightarrow[]{\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \longleftrightarrow R_4} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 \longleftrightarrow R_4} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

So, since we used 2 row interchanges and no row rescalings, we have

$$\begin{aligned}
\det(\mathbf{A}) &= (-1)^2 \det(R.E.F.(\mathbf{A})) \\
&= (-1)^2 \det\left(\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}\right) \\
&= (1)(2)(2)(2) \\
&= 8
\end{aligned}$$