Lecture 27 : Examples of Determinant Computations

Math 3013
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Agenda:

1. Determinants: Results from the preceeding lectures
2. Examples

Determinants

The **determinant** of an $n \times n$ matrix is a polynomial function of the entries of $\mathbf{A}$ with the property that

$$\det(\mathbf{A}) = 0 \quad \Longleftrightarrow \quad \begin{cases} \mathbf{A}^{-1} \text{ does not exist} \\ \mathbf{Ax} = \mathbf{b} \text{ can have more than one solution} \\ \mathbf{Ax} = \mathbf{0} \text{ has solutions other than } \mathbf{x} = \mathbf{0} \\ \mathbf{A} \text{ is } \textbf{not} \text{ row reducible to the identity matrix} \\ \mathit{Rank}(\mathbf{A}) < n \end{cases}$$

Determinant Formulas for Small Matrices ($n \leq 2$)

$\mathbf{n = 1}$:

$$\det\left([a_{11}]\right) = a_{11}$$

$\mathbf{n = 2}$

$$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = a_{11}a_{22} - a_{12}a_{21}$$

Cofactor Expansion Formula for $\det(\mathbf{A})$

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *matrix.*

$$
\begin{aligned}
\det(\mathbf{A}) &= \sum_{j=1}^{n} a_{ij} (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \quad \textit{for each } i = 1, \ldots, n \\
&= \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(\mathbf{M}_{ij}(\mathbf{A})) \quad \textit{for each } j = 1, \ldots, n
\end{aligned}
$$

$\mathbf{M}_{ij} =$ *the* $(n-1) \times (n-1)$ *matrix obtained from* $\mathbf{A}$ *by deleting the row and column containing* a_{ij}.

Example: Determinant of a 4×4 Matrix

Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Then

$$\begin{aligned} \det(\mathbf{A}) &= 0 + 0 + 0 + (3)(-1)^{4+4} \det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}\right) \\ &\quad \text{(cofactor expansion along 4th row)} \\ &= (3)(1)\left(0 + (2)(-1)^{2+2} \det\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right) + 0\right) \\ &\quad \text{(cofactor expansion along 2 row)} \\ &= (3)(2)(1-0) \\ &= 6 \end{aligned}$$

Combinatorial Formula for $\det(\mathbf{A})$

Theorem

$$\det(\mathbf{A}) = \sum_{\substack{permutations \\ \sigma}} sgn(\sigma)\, a_{1,\sigma_1} a_{2,\sigma_2} \cdots a_{n,\sigma_n}$$

This formula shows that

- The determinant of an $n \times n$ matrix $\mathbf{A}$ is a polynomial of degree n in the entries of $\mathbf{A}$.
- Each term of this polynomial has exactly n factors; with each factor coming from a distinct row and distinct column of $\mathbf{A}$.
- In general, this polynomial has $n!$ terms
- This formula, however, while relatively simple to state and interpret, is rarely used for computations

Determinants of Matrices in R.E.F

Theorem

Suppose $\mathbf{A}$ *is in Row Echelon Form (and so upper triangular). Then*

$$\det(\mathbf{A}) = a_{11}a_{22}\cdots a_{nn}$$

(i.e. $\det(\mathbf{A})$ *is just the product of the diagonal entries of* $\mathbf{A}$*).*

Elementary Row Operations and Determinants

Theorem: Let $\mathbf{A}$ be an $n \times n$ matrix and let $\mathcal{R}$ be an elementary row operation.

- If $\mathcal{R}$ is of the type $R_i \longleftrightarrow R_j$ (row interchange)

$$\det\left(\mathcal{R}\left(\mathbf{A}\right)\right) = -\det\left(\mathbf{A}\right)$$

- If $\mathcal{R}$ is of the type $R_i \to \lambda R_i$ (row rescaling)

$$\det\left(\mathcal{R}\left(\mathbf{A}\right)\right) = \lambda \det\left(\mathbf{A}\right)$$

- If $\mathcal{R}$ is of the type $R_i \to R_i + \lambda R_j$

$$\det\left(\mathcal{R}\left(\mathbf{A}\right)\right) = \det\left(\mathbf{A}\right)$$

So while elementary row operations do affect determinants, they only modify them by simple multiplicative factors.

Theorem

Suppose $\mathbf{A}$ *row reduces to a matrix* $\mathbf{A}'$ *in Row Echelon Form. Then*

$$\det(\mathbf{A}) = (-1)^r \frac{1}{\lambda_1 \cdots \lambda_k} \left(a'_{11} a'_{22} \cdots a'_{nn}\right)$$

where

- r *is the number of row interchanges used in the row reduction,*
- $\lambda_1, \ldots, \lambda_k$ *are the row rescaling factors used, and*
- $a'_{11}, \ldots, a'_{nn}$ *are the diagonal elements of the* $\mathbf{A}' = R.E.F.(\mathbf{A})$.

Example

Compute the determinant of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

We have

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 3 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1}} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \longleftrightarrow R_4} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 \longleftrightarrow R_4} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

So, since we used 2 row interchanges and no row rescalings, we have

$$\begin{aligned}
\det(\mathbf{A}) &= (-1)^2 \det(R.E.F.(\mathbf{A})) \\
&= (-1)^2 \det\left(\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}\right) \\
&= (1)(2)(2)(2) \\
&= 8
\end{aligned}$$

Example and Introduction to Eigenvalue Problems

Suppose

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

Determine the values of λ for which the equation

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

has a non-trivial solution (i.e., a solution other than $\mathbf{x} = \mathbf{0}$).

Solution: The stated equation is equivalent to

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

which is a homogeneous linear system with coefficient matrix

$$(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 2 & 1-\lambda \end{bmatrix}$$

Example, Cont'd

Now recall our revised Fundamental Theorem of Invertible Matrices:

Theorem

Suppose $\mathbf{A}$ *is an* $n \times n$ *matrix. Then*

$$\det(\mathbf{A}) = 0 \quad \Longleftrightarrow \quad \begin{cases} \mathbf{A}^{-1} \textit{ does not exist} \\ \mathbf{Ax} = \mathbf{b} \textit{ can have more than one solution} \\ \mathbf{Ax} = \mathbf{0} \textit{ has solutions other than } \mathbf{x} = \mathbf{0} \\ \mathbf{A} \textit{ is } \textbf{not} \textit{ row reducible to the identity matrix} \\ \textit{Rank}(\mathbf{A}) < n \end{cases}$$

By the third statement on the right,

$$(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$$

can have a solution other than $\mathbf{x} = \mathbf{0}$ only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Example, Cont'd

Therefore, to get other, nontrivial, solutions we need

$$\begin{aligned}
0 &= \det\left(\mathbf{A} - \lambda \mathbf{I}\right) \\
&= \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 2 & 2 & 1-\lambda \end{bmatrix}\right) \\
&= (0)(-1)^{1+3}\det\left(\mathbf{M}_{13}\right) + (0)(-1)^{2+3}\det\left(\mathbf{M}_{23}\right) \\
&\quad + (1-\lambda)(-1)^{3+3}\det\left(\mathbf{M}_{33}\right) \\
&= 0 + 0 + (1-\lambda)\left((2-\lambda)^2 - 1\right) \\
&= (1-\lambda)\left(\lambda^2 - 4\lambda + 3\right) \\
&= (1-\lambda)(\lambda-1)(\lambda-3) \\
&= -(\lambda-1)^2(\lambda-3)
\end{aligned}$$

Example, Cont'd

Since we have non-trivial solutions if and only if $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, we must have

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = -(\lambda - 1)^2(\lambda - 3) \quad \Rightarrow \quad \lambda = 1, 3$$

And so we have non-trivial solutions of

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

only if $\lambda = 1, 3$.