Lecture 28: Applications of Determinants

Math 3013
Oklahoma State University

April 6, 2022

Announcement: WebAssign Problem Set 8 should be completed by Monday, April 11.

Agenda:

1. Cramer's Rule
2. The Cofactor Matrix
3. Calculating Matrix Inverses Using Determinants
4. More Examples

Cramer's Rule

Theorem

Suppose $\mathbf{A}$ *is an invertible* $n \times n$ *matrix. Then the components* x_i *of unique solution vector* $\mathbf{x}$ *for the linear system* $\mathbf{A}\mathbf{x} = \mathbf{b}$ *are given by*

$$x_i = \frac{\det(\mathbf{B}_i)}{\det(\mathbf{A})}$$

where $\mathbf{B}_i$ *is the* $n \times n$ *matrix obtained from* $\mathbf{A}$ *by replacing its* i^{th} *column with the column vector* $\mathbf{b}$.

$$\mathbf{B}_i \equiv \begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & \vdots & & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

Cramer's Rule Example

Solve

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ x_1 - x_2 &= -1 \end{aligned}$$

via Cramer's Rule. We have

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

and so

$$\begin{aligned} \det(\mathbf{A}) &= (2)(-1) - (1)(1) = -3 \\ \mathbf{B}_1 &= \begin{bmatrix} 4 & 1 \\ -1 & -1 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{B}_1) = -4 + 1 = -3 \\ \mathbf{B}_2 &= \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{B}_2) = -2 - 4 = -6 \end{aligned}$$

Example, Cont'd

Thus

$$
\begin{aligned}
x_1 &= \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} = \frac{-3}{-3} = 1 \\
x_2 &= \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})} = \frac{-6}{-3} = 2
\end{aligned}
$$

Indeed,

$$
\mathbf{Ax} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \mathbf{b}
$$

Calculating Inverses Using Determinants

Definition

Suppose $\mathbf{A}$ is an $n \times n$ matrix. The $(ij)^{th}$-**cofactor** of $\mathbf{A}$ is the number

$$c_{ij} = (-1)^{i+j} \det(\mathbf{M}_{ij})$$

(Here $\mathbf{M}_{ij}$ is the $(ij)^{th}$-minor of $\mathbf{A}$.)

The numbers c_{ij} , $i = 1, \ldots, n$, $j = 1, \ldots, n$, can be used to construct another $n \times n$ matric $\mathbf{C}$ called the **cofactor matrix C** of $\mathbf{A}$.

Theorem

Suppose $\mathbf{A}$ *is an invertible* $n \times n$ *matrix. Then*

$$\left(\mathbf{A}^{-1}\right)_{ij} = \frac{1}{\det(\mathbf{A})} c_{ji}$$

Alternative Nomenclature

The text (and WebAssign) state this theorem a little differently.

Definition

Let $\mathbf{A}$ be an $n \times n$ matrix. The **matrix adjugate** of $\mathbf{A}$ is the matrix

$$adj\left(\mathbf{A}\right) = \mathbf{C}^t$$

where $\mathbf{C}^t$ is the transpose of the cofactor matrix $\mathbf{C}$ of $\mathbf{A}$.

In terms of the adjugate matrix, we have

Theorem

Let $\mathbf{A}$ be an invertible $n \times n$ matrix, and let $adj\left(\mathbf{A}\right)$ be its adjugate. Then

$$\mathbf{A}^{-1} = \frac{1}{\det\left(\mathbf{A}\right)} adj\left(\mathbf{A}\right)$$

Example: The Inverse of a General 2×2 Matrix

Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We have

$$\begin{aligned} c_{11} &= (-1)^{1+1} \det([d]) = d \\ c_{12} &= (-1)^{1+2} \det([c]) = -c \\ c_{21} &= (-1)^{2+1} \det([b]) = -b \\ c_{22} &= (-1)^{2+2} \det([a]) = a \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{C} &= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \\ \Rightarrow &\quad adj(\mathbf{A}) \equiv \mathbf{C}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ \Rightarrow &\quad \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A}) = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \end{aligned}$$

Indeed,

$$\begin{aligned}
\mathbf{A}^{-1}\mathbf{A} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\
&= \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ca & -cb+ad \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Another Cramer's Rule Example

Solve the following linear system using Cramer's Rule

$$\begin{aligned} x - y + z &= 6 \\ x + 2y &= 0 \\ y + z &= 2 \end{aligned}$$

We have

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}$$

Carrying out a cofactor expansion along the third row of **A** yields

$$\det(\mathbf{A}) = 0 - (1)\det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + (1)\det\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = 4$$

The auxiliary matrices $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ are

$$\mathbf{B}_1 = \begin{bmatrix} 6 & -1 & 1 \\ 0 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \ \mathbf{B}_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \ \mathbf{B}_3 = \begin{bmatrix} 1 & -1 & 6 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

and so

$$\begin{aligned} \det(\mathbf{B}_1) &= -0 + (2)\det\begin{pmatrix} 6 & 1 \\ 2 & 1 \end{pmatrix} - 0 = 8 \\ \det(\mathbf{B}_2) &= -(1)\det\begin{pmatrix} 6 & 1 \\ 2 & 1 \end{pmatrix} + 0 - 0 = -4 \\ \det(\mathbf{B}_3) &= 0 - (1)\det\begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix} + 2\det\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = 12 \end{aligned}$$

Example, Cont'd

And so, the solution vector for the original linear system has the following components

$$\begin{aligned} x &= \frac{\det\left(\mathbf{B}_1\right)}{\det\left(\mathbf{A}\right)} = \frac{8}{4} = 2 \\ y &= \frac{\det\left(\mathbf{B}_2\right)}{\det\left(\mathbf{A}\right)} = \frac{-4}{4} = -1 \\ z &= \frac{\det\left(\mathbf{B}_3\right)}{\det\left(\mathbf{A}\right)} = \frac{12}{4} = 3 \end{aligned}$$

Another Adjugate Matrix and Matrix Inverse Example

Let $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ Find the adjugate matrix and matrix inverse of $\mathbf{A}$.

Let's begin by computing the cofactor matrix of $\mathbf{A}$ entry by entry.

We have

$$
\begin{aligned}
c_{11} &= (-1)^{1+1} \det(\mathbf{M}_{11}) = \det \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} = -1 \\
c_{12} &= (-1)^{1+2} \det(\mathbf{M}_{12}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -1 \\
c_{13} &= (-1)^{1+3} \det(\mathbf{M}_{13}) = \det \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = 1
\end{aligned}
$$

Since we now have all the cofactors for the first row of $\mathbf{A}$, we can readily compute the determinant of $\mathbf{A}$, via a cofactor expansion along the first row:

$$
\begin{aligned}
\det(\mathbf{A}) &= \sum_{j=1}^{3} a_{1j}(-1)^{1+j}\det(\mathbf{M}_{1j}) \\
&= \sum_{j=1}^{3} a_{1j}c_{1j} \\
&= (1)(-1) + (2)(-1) + (1)(1) \\
&= -2
\end{aligned}
$$

But we still have compute the cofactors corresponding to the entries in the second and third rows of $\mathbf{A}$.

Second Row Cofactors:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} c_{21} &= (-1)^{2+1} \det(\mathbf{M}_{21}) = -\det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = -1 \\ c_{22} &= (-1)^{2+2} \det(\mathbf{M}_{22}) = \det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1 \\ c_{23} &= (-1)^{2+3} \det(\mathbf{M}_{23}) = -\det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = -1 \end{aligned}$$

Third Row Cofactors:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} c_{31} &= (-1)^{3+1} \det(\mathbf{M}_{31}) = \det\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} = 1 \\ c_{32} &= (-1)^{3+2} \det(\mathbf{M}_{32}) = -\det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = 1 \\ c_{33} &= (-1)^{3+3} \det(\mathbf{M}_{33}) = \det\begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = -3 \end{aligned}$$

We have now computed all the cofactors c_{ij} for $\mathbf{A}$.
These can now be arranged to form the **cofactor matrix**

$$\mathbf{C} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & 1 & -3 \end{bmatrix}$$

The adjunct matrix of $\mathbf{A}$ is then

$$adj\,(\mathbf{A}) = \mathbf{C}^t = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & -3 \end{bmatrix}$$

and so

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj\,(\mathbf{A}) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$