Lecture 29 : The Eigenvalue-Eigenvector Problem

Math 3013
Oklahoma State University

April 11, 2022

Agenda:

1. The Eigenvalue-Eigenvector Problem
2. Step 1: Determining the Eigenvalues of a Matrix
3. Step 2: Determining the Eigenspace for Each Eigenvalue of a Matrix

Eigenvalues and Eigenvectors

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Definition

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Consider

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad , \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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and so $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 4.

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Definition

Let λ be an eigenvalue of an $n \times n$ matrix $\mathbf{A}$. The λ-**eigenspace** of $\mathbf{A}$ is

$$E_\lambda \equiv \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{v} = \lambda\mathbf{v}\}$$

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we see that finding the eigenvectors with eigenvalue λ is equivalent to solving the homogeneous linear system

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&\equiv \mathit{NullSp}\,(\mathbf{A} - \lambda\mathbf{I}) \\
&= \mathit{NullSp}\,(\mathbf{A}_\lambda)
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Lemma

If λ is an eigenvalue of an $n \times n$ matrix $\mathbf{A}$, then E_λ is a subspace of $\mathbb{R}^n$.

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Step 1: Find the eigenvalues of $\mathbf{A}$

Step 2: For each eigenvalue λ find the solutions of

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0}$$

(by row-reducing $(\mathbf{A} - \lambda \mathbf{I})$ to R.R.E.F.)

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- However, eigenvectors are, by definition, **non-trivial solutions** of (1).
- The real question is thus: when do we have solutions of (1) other than $\mathbf{x} = \mathbf{0}$.

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Suppose $\mathbf{A}$ *is an* $n \times n$ *matrix. Then the following statements are equivalent:*

(i) $\mathbf{A}$ *has a matrix inverse.*

(ii) *Every linear system of the form* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*

(iii) $\mathbf{x} = \mathbf{0}$ *is the unique solution to* $\mathbf{Ax} = \mathbf{0}$.

(iv) $R.R.E.F.(\mathbf{A}) = \mathbf{I}$

(v) $\det(\mathbf{A}) \neq 0$

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Finding Nontrivial Solutions, Cont'd

Negating both statements (iii) and (v),
Statement (iii) $\rightarrow$ $\mathbf{Ax} = \mathbf{0}$ has solutions other than $\mathbf{x} = \mathbf{0}$
Statement (v) $\rightarrow$ $\det(\mathbf{A}) = 0$
We conclude

Corollary

A homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ *has nontrivial solutions*

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Corollary

A homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ *has nontrivial solutions (i.e. solutions other than* $\mathbf{x} = \mathbf{0}$*)* **if and only if**

$$\det(\mathbf{A}) = 0$$

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$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \qquad (*)$$

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Since the determinant of a matrix is a polynomial in entries of the matrix, the condition (*) can be regarded as a polynomial equation for the eigenvalue λ.

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$$p_{\mathbf{A}}\left(\lambda\right) \equiv \det\left(\mathbf{A} - \lambda\mathbf{I}\right)$$

is a polynomial of degree n in the parameter λ.

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The Characteristic Equation for a matrix **A**

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is a polynomial of degree n in the parameter λ. $p_{\mathbf{A}}(\lambda)$ is called the **characteristic polynomial** of **A** and the polynomial equation

$$p_{\mathbf{A}}(\lambda) = 0$$

is called the **characteristic equation** of **A**.

Finding the Eigenvalues of an $n \times n$ matrix $\mathbf{A}$

The eigenvalues of an $n \times n$ matrix $\mathbf{A}$ can be found by calculating its characteristic polynomial

$$p_{\mathbf{A}}(\lambda) \equiv \det(\mathbf{A} - \lambda \mathbf{I})$$

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for λ.
In other words, the solutions of the characteristic equation are exactly the eigenvalues of $\mathbf{A}$

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$$\det(\mathbf{A} - \lambda \mathbf{I})$$

is always a polynomial of degree n in λ.

Definition

Let $\mathbf{A}$ be an $n \times n$ matrix. The **characteristic polynomial** of $\mathbf{A}$ is the polynomial

$$p_{\mathbf{A}}(\lambda) \equiv \det(\mathbf{A} - \lambda \mathbf{I})$$

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The Corollary then tells us that if λ is eigenvalue of $\mathbf{A}$, then λ is a root (i.e. a solution) of $p_{\mathbf{A}}(\lambda) = 0$.
We thus find the eigenvalues of $\mathbf{A}$ by finding all the roots of $p_{\mathbf{A}}(\lambda) = 0$.

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Therefore, the eigenvalues of $\mathbf{A}$ are 4 and -2.

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Thus, we solve a polynomial equation $p(x) = 0$ by factorizing $p(x)$.

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Right away we see $\lambda = (\lambda - 0)$ divides $p_{\mathbf{A}}(\lambda)$; and so $\lambda = 0$ must be a root.

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We conclude that the eigenvalues of $\mathbf{A}$ are 0, 2 and -1.

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- Grab the basis vectors for the solution set. (These will be your basic eigenvectors for the given eigenvalue)