# Lecture 30: Eigenvalues, Eigenvectors, and Multiplicities

Math 3013
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1. The Eigenvalue/Eigenvector Problem
2. Finding the Eigenvalues
3. Finding the Eigenvectors
4. Algebraic and Geometric Multiplicities

# The Eigenvalue/Eigenvector Problem

Given an $n \times n$ matrix $\mathbf{A}$, find all the numbers $\lambda$ such that there is a **non-zero** vectors $\mathbf{v}$ such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{1}$$

When we find such a solution $\lambda, \mathbf{v}$ to the eigenvalue/eigenvector problem, $\lambda$ is called an **eigenvalue** of $\mathbf{A}$ and $\mathbf{v}$ is called an **eigenvector** of $\mathbf{A}$.

Note that the zero vector $\mathbf{0}$ is always a solution of (1)

However, the solution $\mathbf{v} = \mathbf{0}$ is **irrelevant** to the EV/EV Problem.

# The EV/EV Condition as a Homogeneous Linear System

The equation

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v} \tag{1}$$

is equivalent to the homogeneous linear system

$$(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0} \tag{2}$$

with $(\mathbf{A} - \lambda \mathbf{I})$ as its coefficient matrix.
Since we are looking for non-zero solutions, we need (2) to have more than one solution; and this, in turn requires the coefficient matrix $(\mathbf{A} - \lambda \mathbf{I})$ to satisfy

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{3}$$

(Recall $\det(\mathbf{M}) \neq 0 \quad \Longleftrightarrow \quad \mathbf{M}\mathbf{x} = \mathbf{0}$ has only one solution, $\mathbf{x} = \mathbf{0}$).

# The Characteristic Polynomial and Characteristic Equation

The determinant of an $n \times n$ matrix is always a homogeneous polynomial degree $n$ in the entries of the matrix. For this reason,

$$p_{\mathbf{A}}(\lambda) \equiv \det(\mathbf{A} - \lambda \mathbf{I})$$

is a polynomial of degree $n$ in $\lambda$.
We call $p_{\mathbf{A}}(\lambda)$ the **characteristic polynomial** of **A**.

Thus, a necessary condition for solving the EV/EV Problem is that $\lambda$ satisfy the polynomial equation

$$p_{\mathbf{A}}(\lambda) = 0 \tag{1}$$

This equation is called the **characteristic equation** of the matrix **A**.

# Some basic facts about polynomial equations

- If $p(x)$ is a polynomial of degree $n$ then there are at most $n$ solutions of $p(x) = 0$.
- Every polynomial of degree $n$ has a factorization of the form

$$p(x) = a(x - r_1)(x - r_2)\cdots(x - r_n) \quad , \quad a, r_1, \ldots, r_n \in \mathbb{C} \tag{4}$$

- $x = r$ is a solution of $p(x) = 0$ if and only if $(x - r)$ occurs as the factor on the right hand side of (4)

Thus, a good strategy for solving

$$p_{\mathbf{A}}(\lambda) = 0$$

would be to find the factorization

$$p_{\mathbf{A}}(\lambda) = a(\lambda - r_1)\cdots(\lambda - r_n)$$

(the solutions will then be $\lambda = r_1, r_2, \ldots, r_n$).

# Example 1

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Find all the eigenvalues of $\mathbf{A}$. We need to solve

$$\begin{aligned} 0 &= p_{\mathbf{A}}(\lambda) \\ &\equiv \det(\mathbf{A} - \lambda \mathbf{I}) \\ &= \det \begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} \\ &= (1-\lambda)(2-\lambda)(1-\lambda) \end{aligned}$$

The only solutions of this equation are $\lambda = 1$ and $\lambda = 2$. And so the eigenvalues of $\mathbf{A}$ are the numbers 1 and 2.

# Roots and Multiplicities

In the preceding example $p_{\mathbf{A}}(\lambda)$ had three factors but only two distinct eigenvalues. This turns out to be a special property of $p_{\mathbf{A}}(\lambda)$ for which we have some special nomenclature.

### Definition

Suppose a polynomial $p(x)$ factorizes as

$$p(x) = a(\lambda - r_1)^{m_1}(\lambda - r_2)^{m_2} \cdots (\lambda - r_k)^{m_k}$$

The numbers $r_1, \ldots, r_n$ are called the **roots** of $p(x)$ and the integers $m_i$ , $i = 1, \ldots, k$, are called the **multiplicities** of the roots $r_i$.

# Eigenvalues and their Algebraic Multiplicities

In Linear Algebra, we use the following, more specialized, terminolgy:

The solutions of

$$p_{\mathbf{A}}(\lambda) = 0$$

are the **eigenvalues** of **A** and the number of times $(\lambda - r)$ appears as a factor of $p_{\mathbf{A}}(\lambda)$ is called the **algebraic multiplicity** of the eigenvalue $r$.

We'll soon see that there is another kind of "multiplicity" associated with eigenvalues.

# Example 1 Cont'd

Thus, in the preceding example, where we found

$$p_{\mathbf{A}}(\lambda) = (1-\lambda)(2-\lambda)(1-\lambda) = (1-\lambda)^2(2-\lambda)$$

we would say the matrix **A** has two eigenvalues:

$$\begin{aligned} \lambda &= 1 \quad \text{with algebraic multiplicity } 2 \\ \lambda &= 2 \quad \text{with algebraic multiplicity } 1 \end{aligned}$$

# Finding the Eigenvectors

Once we have found all the solutions of

$$0 = p_{\mathbf{A}}(\lambda) \equiv \det(\mathbf{A} - \lambda \mathbf{I})$$

we have found the eigenvalues of $\mathbf{A}$: i.e., we have solved the first half of the eigenvalue/eigenvector problem.

The next step is to figure out for each eigenvalue $\lambda = r$ of $\mathbf{A}$, the corresponding eigenvectors.

**Nomenclature:** The set

$$E_r = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{v} = r\mathbf{v}\}$$

is called the $r$-**eigenspace** of $\mathbf{A}$. $E_r$ is a subspace of $\mathbb{R}^n$ Indeed,

$$E_r = NullSp(\mathbf{A} - r\mathbf{I})$$

and so we find the eigenvectors in $E_r$ by solving

$$(\mathbf{A} - r\mathbf{I})\,\mathbf{x} = 0$$

# Eigenvectors and Geometric Multiplicities

As a non-zero subspace of $\mathbb{R}^n$, $E_r$ will contain **infinitely** many vectors.

OTOH, as a subspace of $\mathbb{R}^n$, it will have a **finite** basis.

Abusing our language a bit, when we find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for $E_r$, we say that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for $E_r$ are **the** eigenvectors of $\mathbf{A}$ with eigenvalue $r$.

Finding a basis for each non-zero $E_r$ is how the solution of the EV/EV Problem is completed.

**Nomenclature:** The **geometric multiplicity** of an eigenvalue $r$ is the dimension of the $r$-eigenspace $E_r$ (i.e. the number of vectors in any basis for $E_r$).

# Summary of EV/EV Nomenclature:

Given an $n \times n$ matrix $\mathbf{A}$

- $p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I})$, the **characteristic polynomial** of $\mathbf{A}$
- When $p_{\mathbf{A}}(\lambda)$ is factorized

$$p_{\mathbf{A}}(\lambda) = (\lambda - r_1)^{m_1} (\lambda - r_2)^{m_2} \cdots (\lambda - r_k)^{m_k} \qquad (*)$$

  the numbers $r_1, \ldots, r_k$ are the **eigenvalues** of $\mathbf{A}$
- For any eigenvalue $r_i$ of $\mathbf{A}$
  - the integer power $m_i$ occuring in the factorization (*) is the **algebraic multiplicity** of the eigenvalue $r_i$
  - the subspace

$$E_{r_i} = NullSp(\mathbf{A} - r_i \mathbf{I}) = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{v} = r_i \mathbf{v}\}$$

    is the $r_i$-**eigenspace** of $\mathbf{A}$.
  - we call any basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\}$ for $E_{r_i}$ the **eigenvectors of A with eigenvalue** $r_i$
  - the number $\ell$ of basis vectors for $E_{r_i}$ is called the **geometric multiplicity** $\mu_i$ of the eigenvalue $r_i$

$$\mu_i = \dim(E_{r_i}) = Nullity(\mathbf{A} - r_i \mathbf{I})$$

# Properties of Algebraic and Geometric Multiplicities

Suppose:
$r_1, \ldots, r_k$ are the eigenvalues of an $n \times n$ matrix $\mathbf{A}$
$m_1, \ldots, m_k$ are the corresponding algebraic multiplicities ($m_i =$ the number of factors of $(\lambda - r_i)$ in $p_{\mathbf{A}}(\lambda)$)
$\mu_1, \ldots, \mu_k$ are the corresponding geometric multiplities ($\mu_i = \dim(E_{r_i})$)
Then

- $$\sum_{i=1}^{k} m_i = n$$

- $$\mu_i \leq m_i \text{ for all } i$$

# Example 1 Cont'd

Let's now find the eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

This the same matrix as before and so we already know that its eigenvalues are $\lambda = 1$ and $\lambda = 2$.

## $E_1$ ; the 1-eigenspace of $\mathbf{A}$

We have

$$\begin{aligned} E_1 &= NullSp\left(\mathbf{A} - (1)\,\mathbf{I}\right) = NullSp\begin{pmatrix} 1-(1) & 1 & 0 \\ 0 & 2-(1) & 1 \\ 0 & 0 & 1-(1) \end{pmatrix} \\ &= NullSp\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= NullSp\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The last matrix is the R.R.E.F. of $\left(\mathbf{A} - (1)\,\mathbf{I}\right)$. Thus, if $\left(\mathbf{A} - (1)\,\mathbf{I}\right)\mathbf{x} = \mathbf{0}$, we must have

$$\left.\begin{aligned} x_2 &= 0 \\ x_3 &= 0 \\ 0 &= 0 \end{aligned}\right\}$$

and $x_1$ is a free variable in the solution.

# $E_1$ ; the 1-eigenspace of $\mathbf{A}$, Cont'd

Thus, a solution vector must have the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The eigenvector corresponding to the eigenvalue 1 is thus

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Since we have only one basis vector for the 1-eigenspace, $E_1$, the geometric multiplicity of the eigenvalue 1 is 1.

## $E_2$ ; the 2-eigenspace of $\mathbf{A}$

We need to determine

$$\begin{aligned} NullSp\left(\mathbf{A}-(2)\,\mathbf{I}\right) &= NullSp\begin{pmatrix} 1-(2) & 1 & 0 \\ 0 & 2-(2) & 1 \\ 0 & 0 & 1-(2) \end{pmatrix} \\ &= NullSp\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \\ &= NullSp\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and so we need

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_3 &= 0 \\ 0 &= 0 \end{aligned}\right\} \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

## $E_2$ ; the 2-eigenspace of $\mathbf{A}$, Cont'd

Hence, a basis vector for $E_2$ will be

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and the geometric multiplicity of the eigenvalue $\lambda = 2$ is 1.

# Example 1: Summary

The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

has

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = (1-\lambda)^2 (2-\lambda)$$

as its characteristic polynomial and we have the following table

| eigenvalue | eigenvectors | alg. mult. | geo. mult |
|---|---|---|---|
| 1 | $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ | 2 | 1 |
| 2 | $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ | 1 | 1 |

# Example 2

Find the eigenvalues and eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ -7 & 2 & -1 \\ 7 & 0 & 2 \end{bmatrix}$$

and then determine the algebraic and geometric multiplicities of each eigenvalue.

$$p_{\mathbf{A}}(\lambda) = \det \begin{pmatrix} -4-\lambda & 0 & 0 \\ -7 & 2-\lambda & -1 \\ 7 & 0 & 2-\lambda \end{pmatrix} = (-4-\lambda)(2-\lambda)^2$$

We thus have 2 eigenvalues $\lambda = -4, 2$. The eigenvalue $\lambda = -4$ has algebraic multiplicity 1 (since there is 1 factor of $(-4-\lambda)$ in $p_{\mathbf{A}}(\lambda)$), while the eigenvalue $\lambda = 2$ has algebraic multiplicity 2 (since there are 2 factors of $(2-\lambda)$ in $p_{\mathbf{A}}(\lambda)$).

# Example 2: $-4$-eigenspace

$$NullSp\left(\mathbf{A} - (-4)\,\mathbf{I}\right) = NullSp\begin{pmatrix} 0 & 0 & 0 \\ -7 & 6 & -1 \\ 7 & 0 & 6 \end{pmatrix} = NullSp\begin{pmatrix} 1 & 0 & \frac{6}{7} \\ 0 & 1 & \frac{5}{7} \\ 0 & 0 & 0 \end{pmatrix}$$

The vector $\mathbf{v}_{\lambda=-4} = \begin{bmatrix} -\frac{6}{7} \\ -\frac{5}{7} \\ 1 \end{bmatrix}$ thus provides a basis for the $\lambda = -4$ eigenspace. Since the $(-4)$-eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue $-4$ is 1.

# Example 2: 2-eigenspace

$$\begin{aligned} NullSp\left(\mathbf{A} - (2)\,\mathbf{I}\right) &= NullSp\begin{pmatrix} -6 & 0 & 0 \\ -7 & 0 & -1 \\ 7 & 0 & 0 \end{pmatrix} \\ &= NullSp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= span\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \end{aligned}$$

The vector $\mathbf{v}_{\lambda=2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ thus provides a basis for the $\lambda = 2$ eigenspace. Since the eigenspace is 1-dimensional, the geometric multiplicity of the eigenvalue 2 is 1.

# Example 2: Summary

The matrix

$$\mathbf{A} = \begin{bmatrix} -4 & 0 & 0 \\ -7 & 2 & -1 \\ 7 & 0 & 2 \end{bmatrix}$$

has

$$p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = (-4 - \lambda)(2 - \lambda)^2$$

as its characteristic polynomial and we have the following table of multiplicities

| eigenvalue | eigenvectors | alg. mult. | geo. mult |
|---|---|---|---|
| $-4$ | $\left\{ \begin{bmatrix} -\frac{6}{7} \\ -\frac{5}{7} \\ 1 \end{bmatrix} \right\}$ | 1 | 1 |
| 2 | $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ | 2 | 1 |