Lecture 31: Diagonalization of Square Matrices

Math 3013
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Agenda

1. Diagonal Matrices
2. Diagonalizability
3. Some Matrix Multiplication Identities
4. Connection with Eigenvalue Multiplicities
5. Other Special Cases for Diagonalizability

Diagonal Matrices

Definition

An $n \times n$ matrix is **diagonal** if its only non-zero entries occur along its main diagonal; i.e.

$$(\mathbf{A})_{ij} = 0 \quad \text{if} \quad i \neq j$$

In other words, a diagonal matrix is a matrix of the form

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

Two Special Properties of Diagonal Matrices

Let $\mathbf{D}$, $\mathbf{D}'$ be a diagonal matrices :
Say

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \quad , \quad \mathbf{D}' = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu_n \end{bmatrix}$$

Then

$$\mathbf{D}\mathbf{D}' = \begin{bmatrix} \lambda_1\mu_1 & 0 & \cdots & 0 \\ 0 & \lambda_2\mu_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n\mu_n \end{bmatrix} \tag{1}$$

Two Special Properties of Diagonal Matrices, Cont'd

Suppose $\mathbf{e}_i \in \mathbb{R}^n$ is the i^{th} standard basis vector. Then

$$\mathbf{D}\mathbf{e}_i = \lambda_i \mathbf{e}_i \qquad (*)$$

i.e., $\mathbf{e}_i$ is an eigenvector of $\mathbf{D}$ with eigenvalue λ_i.

N.B. Property (*) makes it easy to compute the action of diagonal matrices on more general vectors. If $\mathbf{v} = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^n$

$$\begin{aligned}
\mathbf{D}\mathbf{v} &= \mathbf{D}\left(v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n\right) \\
&= v_1\left(\mathbf{D}\mathbf{e}_1\right) + v_2\left(\mathbf{D}\mathbf{e}_2\right) + \cdots + v_n\left(\mathbf{D}\mathbf{e}_n\right) \\
&= \left(v_1\lambda_1\right)\mathbf{e}_1 + \left(v_2\lambda_2\right)\mathbf{e}_2 + \cdots + \left(v_n\lambda_n\right)\mathbf{e}_n
\end{aligned}$$

Thus, even if $\mathbf{v}$ is not an eigenvector of $\mathbf{D}$, the action of $\mathbf{D}$ on $\mathbf{v}$ is relatively simple to compute (and understand).

Moral: Dealing with a matrix $\mathbf{A}$ is much easier when the matrix is diagonal

Diagonalizability

Definition

An $n \times n$ matrix $\mathbf{A}$ is said to be **diagonalizable** if there is an **invertible** $n \times n$ matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$$

Today, I'll explain how to find the matrix $\mathbf{C}$ that will convert an $n \times n$ matrix $\mathbf{A}$ into a diagonal matrix $\mathbf{A}$.
In particular, I show how the "diagonalizing matrix" $\mathbf{C}$ can be constructed from the eigenvectors of $\mathbf{A}$ and how the corresponding diagonal matrix $\mathbf{D}$ is constructed from the eigenvalues of $\mathbf{A}$.

Digression: Some Matrix Multiplication Identities

Lemma

Let $\mathbf{A}$ *and* $\mathbf{C}$ *are* $n \times n$ *matrices, and suppose* $\mathbf{c}_1, \ldots, \mathbf{c}_n$ *are column vectors of* $\mathbf{C}$*. then*

$$\mathbf{AC} = \begin{bmatrix} \uparrow & & \uparrow \\ (\mathbf{Ac}_1) & \cdots & (\mathbf{Ac}_n) \\ \downarrow & & \downarrow \end{bmatrix}$$

Example

$$\begin{aligned} \mathbf{AC} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix} \end{aligned}$$

Meanwhile

$$\begin{aligned} \mathbf{Ac}_1 &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} \\ a_{21}c_{11} + a_{22}c_{21} \end{bmatrix} \\ \mathbf{Ac}_2 &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} = \begin{bmatrix} a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{12} + a_{22}c_{22} \end{bmatrix} \end{aligned}$$

Digression: Some Matrix Multiplication Identities, Cont'd

Lemma

Suppose $\mathbf{C}$ *is an* $n \times n$ *matrix with columns* $\mathbf{c}_1, \ldots, \mathbf{c}_n$ *and* $\mathbf{D}$ *is a diagonal* $n \times n$ *matrix*

$$\mathbf{C} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{c}_1 & \cdots & \mathbf{c}_n \\ \downarrow & & \downarrow \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Then

$$\mathbf{CD} = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ (\lambda_1 \mathbf{c}_1) & \cdots & (\lambda_n \mathbf{c}_n) \\ \downarrow & \cdots & \downarrow \end{bmatrix}$$

Proof of Lemma

Write

$$(D)_{ij} = \lambda_i \delta_{ij} \quad , \quad \text{where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then

$$\begin{aligned} (\mathbf{CD})_{ij} &= \sum_{k=1}^{n} c_{ik} d_{kj} \\ &= \sum_{k=1}^{n} (\mathbf{c}_k)_i (\lambda_k \delta_{jk}) \\ &= (\mathbf{c}_j)_i \lambda_j \end{aligned}$$

So the j^{th} column of **CD** is

$$Col_j(\mathbf{CD}) = \begin{bmatrix} \uparrow \\ \lambda_j \mathbf{c}_j \\ \downarrow \end{bmatrix}$$

A Necessary Condition for Diagonalizability

Theorem

Suppose $\mathbf{A}$ *is a diagonalizable* $n \times n$ *matrix; so there is an* $n \times n$ *matrix* $\mathbf{C}$ *and a* $n \times n$ *diagonal matrix* $\mathbf{D}$ *such that* $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$*. Write*

$$\mathbf{C} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{c}_1 & \cdots & \mathbf{c}_n \\ \downarrow & & \downarrow \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

Then for each $i = 1, \ldots, n$*, the* i^{th} *column,* $\mathbf{c}_i$*, of* $\mathbf{C}$ *must be an eigenvector of* $\mathbf{A}$ *with eigenvalue* $\lambda_i = i^{th}$ *diagonal entry of* $\mathbf{D}$*.*

Proof of Theorem

Multiplying both sides of the equation $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$ from the left by $\mathbf{C}$ we get

$$\mathbf{A}\mathbf{C} = \mathbf{C}\mathbf{D}$$

Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ be the columns of $\mathbf{C}$. From our matrix multiplication identities

$$\mathbf{A}\mathbf{C} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{A}\mathbf{c}_1 & \cdots & \mathbf{A}\mathbf{c}_n \\ \downarrow & & \downarrow \end{bmatrix} \quad , \quad \mathbf{C}\mathbf{D} = \begin{bmatrix} \uparrow & & \uparrow \\ \lambda_1\mathbf{c}_1 & \cdots & \lambda_n\mathbf{c}_n \\ \downarrow & & \downarrow \end{bmatrix}$$

and the Theorem follows by comparing $\mathbf{A}\mathbf{C}$ and $\mathbf{C}\mathbf{D}$ column by column. □

The Diagonalization Algorithm

The theorem suggests how we might find a matrix **C** that diagonalizes **A**:

(i) Find n eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of **A** and use them as the columns of a matrix **C** (in the same order)

(ii) Use the eigenvalue of $\mathbf{v}_i$ as the i^{th} entry of a diagonal matrix **D**

We then have

$$\mathbf{C} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

and so, by the theorem,

$$\mathbf{AC} = \mathbf{CD}$$

If C is invertible, we can then multiply both sides of this last equation by $\mathbf{C}^{-1}$ to get

$$\mathbf{C}^{-1}\mathbf{AC} = \mathbf{D}$$

thus diagonalizing **A**.

Example

Find a matrix **C** that diagonalizes $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$

Step 1: We first need to find the eigenvalues of **A**.

$$\begin{aligned} 0 &= p_{\mathbf{A}}(\lambda) = \det \begin{pmatrix} 2-\lambda & 6 \\ 0 & -1-\lambda \end{pmatrix} = (2-\lambda)(-1-\lambda) \\ \Rightarrow & \quad \lambda = 2, -1 \end{aligned}$$

E_2 : 2-eigenspace

Step 2: Find the eigenvectors of **A**

$$\begin{aligned} E_2 &= NullSp\left(\begin{bmatrix} 2-2 & 6 \\ 0 & -1-2 \end{bmatrix}\right) \\ &= NullSp\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \\ \Rightarrow & \quad x_2 = 0 \text{ , } x_1 \text{ is free} \\ \Rightarrow & \quad \mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \Rightarrow & \quad \mathbf{v}_{\lambda=2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

So

$$E_2 = span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

E_{-1} : (-1)-eigenspace

$$
\begin{aligned}
E_{-1} &= NullSp\left(\begin{bmatrix} 2-(-1) & 6 \\ 0 & -1-(-1) \end{bmatrix}\right) \\
&= NullSp\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) \\
&\Rightarrow \quad x_1 = -2x_2 \quad , \quad x_2 \text{ is free} \\
&\Rightarrow \quad \mathbf{x} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\
\mathbf{v}_{\lambda=-1} &= \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\end{aligned}
$$

Example, Cont'd

We thus have two linearly independent eigenvectors and so we can form an invertible matrix $\mathbf{C}$ using $\mathbf{v}_{\lambda=2}$ and $\mathbf{v}_{\lambda=-1}$ as columns

$$\mathbf{C} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Using the cofactor formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

we find

$$\mathbf{C}^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Example, Cont'd

And so

$$\begin{aligned}\mathbf{C}^{-1}\mathbf{AC} &= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 4 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\end{aligned}$$

Note that the diagonal entries of the diagonal matrix are just the eigenvalues of **A** (the first eigenvalue 2 corresponding to the first eigenvector $\mathbf{v}_{\lambda=2}$ and the second eigenvalue -1 corresponding to the second eigenvector $\mathbf{v}_{\lambda=-1}$).

How to ensure that $\mathbf{C}$ is invertible

We have just seen that we can diagonalize $\mathbf{A}$ if we can form an **invertible** matrix $\mathbf{C}$ using the eigenvectors of $\mathbf{A}$ as columns. Recall an $n \times n$ matrix $\mathbf{C}$ is invertible if and only if each of the following statements are true

(i) $\mathbf{C}$ is row reducible to the identity matrix

(ii) $\mathbf{Cx} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$

(iii) The only solution of $\mathbf{Cx} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$

(iv) $rank\left(\mathbf{C}\right) = n$

(v) $\det\left(\mathbf{C}\right) \neq 0$

Since $rank\left(\mathbf{C}\right) = \dim\left(ColSp\left(\mathbf{C}\right)\right)$, (iv) in turn requires that the columns of $\mathbf{C}$ are linearly independent.

Thus,

Corollary

An $n \times n$ *matrix* $\mathbf{A}$ *is diagonalizable* **if and only if** *it has* n *linearly independent eigenvectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$. *In this situation, if we set*

$$\mathbf{C} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

with

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$$

Then $\mathbf{C}$ *is invertible and*

$$\mathbf{AC} = \mathbf{CD} \implies \mathbf{C}^{-1}\mathbf{AC} = \mathbf{D}$$

Lemma

Suppose $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are eigenvectors of a matrix* $\mathbf{A}$ *with different eigenvalues:*

$$\begin{aligned}\mathbf{A}\mathbf{v}_1 &= \lambda_1 \mathbf{v}_1 \\ \mathbf{A}\mathbf{v}_2 &= \lambda_2 \mathbf{v}_2\end{aligned}$$

with $\lambda_1 \neq \lambda_2$. *Then* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *are linearly independent.*

Proof

Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ are not linearly independent. Then there are non-zero numbers x_1 and x_2 such that

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{0} \quad \Rightarrow \quad \mathbf{v}_1 = -\frac{x_2}{x_1}\mathbf{v}_2$$

But then

$$\begin{aligned}
\lambda_1\mathbf{v}_1 &= \mathbf{A}\mathbf{v}_1 \\
&= \mathbf{A}\left(-\frac{x_2}{x_1}\mathbf{v}_2\right) \\
&= \left(-\frac{x_2}{x_1}\right)\mathbf{A}\mathbf{v}_2 \\
&= \left(-\frac{x_2}{x_1}\right)(\lambda_2\mathbf{v}_2) \\
&= \lambda_2\left(-\frac{x_2}{x_1}\mathbf{v}_2\right) \\
&= \lambda_2\mathbf{v}_1
\end{aligned}$$

More generally

which implies

$$(\lambda_1 - \lambda_2)\, \mathbf{v}_2 = \mathbf{0}$$

which can't happen since $\lambda_1 \neq \lambda_2$ and $\mathbf{v}_2$ is a non-zero vector.

Theorem

If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors of a matrix $\mathbf{A}$ with different eigenvalues, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ are linearly independent.

So if an $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues, it will be diagonalizable.

Diagonalizability and Multiplicities of Eigenvalues

We have seen that an $n \times n$ matrix is diagonalizable if and only if it has n linearly independent eigenvectors.
I'll now show you another criterion for diagonalizability based on the notions of algebraic and geometric multiplicities.
Recall that the **algebraic multiplicity** of an eigenvalue r of an $n \times n$ matrix $\mathbf{A}$ is the number of factors of $(\lambda - r)$ that occur in the characteristic polynomial $p_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$ of $\mathbf{A}$. If

$$p_{\mathbf{A}}(\lambda) = a(\lambda - r_1)^{m_1} \cdots (\lambda - r_k)^{m_k}$$

then

$$Mult_{alg}(r_i) = m_i$$

Note that since $\deg(p_{\mathbf{A}}(\lambda)) = n$, we always have

$$n = m_1 + m_2 + \cdots + m_k$$

Diagonalizability and Multiplicities of Eigenvalues, Cont'd

The **geometric multiplicity** μ of an eigenvalue r of $\mathbf{A}$ is the dimension of the r-eigenspace of $\mathbf{A}$:

$$\mu = Mult_{geom}(r) = \dim(NullSp(\mathbf{A} - r\mathbf{I}))$$

The geometric multiplicities effectively counts the number of linearly independent eigenvectors in each eigenspace. If we sum over the geometric multiplicities μ_i of each eigenvalue r_i, we get the total number of linearly independent eigenvectors. Thus,

$$\#\text{ linearly independent eigenvectors } = \mu_1 + \mu_2 + \cdots + \mu_k$$

Thus,

Theorem

An $n \times n$ matrix is diagonalizable if and only if the geometric multiplicities of its eigenvalues sum to n.

$$n = \mu_1 + \cdots + \mu_k \qquad \Longleftrightarrow \qquad \mathbf{A} \textit{ is diagonalizable}$$

Numerics for Algebraic and Geometric Multiplicities

Theorem

Let $r_1, \ldots, r_k$ be the eigenvalues of an $n \times n$ matrix $\mathbf{A}$, let $m_1, \ldots, m_k$ be the corresponding algebraic multiplicities, and let $\mu_1, \ldots, \mu_k$ be the corresponding geometric multiplicities. Then

$$1 \leq \mu_i \leq m_i \text{ for } i = 1, \ldots, k$$

Corollary

Let $r_1, \ldots, r_k$ be the eigenvalues of an $n \times n$ matrix $\mathbf{A}$. Then

$$\mathbf{A} \text{ is diagonalizable} \quad \Longleftrightarrow \quad Mult_{geom}(r_i) = Mult_{alg}(r_i) \text{ for all } i$$

Put another way, **A** is **not diagonalizable** if there is an eigenvalue r_i such that

$$Mult_{geom}(r_i) < Mult_{alg}(r_i)$$

For if $m_i = Mult_{alg}(r_i)$ and $\mu_i = Mult_{geom}(r_i)$, then

$$\begin{aligned} \#\text{ lin. indep. eigenvectors} &= \mu_1 + \cdots + \mu_k \\ &< m_1 + \cdots + m_k = n \end{aligned}$$

Example

Determine if the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is diagonalizable.

We have

$$0 = p_{\mathbf{A}}(\lambda) = \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$$

and so we have one eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

$$E_1 = NullSp \begin{pmatrix} 1-1 & 1 \\ 0 & 1-1 \end{pmatrix} = NullSp \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = span\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

Since E_1 is 1-dimensional, the geometric multiplicity of $\lambda = 1$ is 1.
Since

$$1 = \text{Geo. Mult.}\,(\lambda = 1) < \text{Alg. Mult.}\ (\lambda = 1) = 2$$

The matrix **A** is **not diagonalizable**.

Other Criteria for Diagonalizability

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *matrix.*

(i) $\mathbf{A}$ *has* n *distinct eigenvalues, then* $\mathbf{A}$ *is diagonalizable.*

(ii) *If* $\mu_i = m_i$ *for each eigenvalue* r_i, $\mathbf{A}$ *is diagonalizable.*

(iii) *If* $\mathbf{A}$ *has an eigenvalue* r_i *for which the corresponding geometric multiplicity* μ_i *is strictly less than the corresponding algebraic multiplicity* m_i, *then* $\mathbf{A}$ *is not diagonalizable.*

Other Tests for Diagonalizability, Cont'd

Lastly, we have the following theorem which is frequently applicable in physical problems.

Theorem

If $\mathbf{A}$ *is a symmetric matrix (i.e.,* $\mathbf{A}^t = \mathbf{A}$*), then*

(i) *All the eigenvalues of* $\mathbf{A}$ *are real numbers.*

(ii) $\mathbf{A}$ *is diagonalizable.*