# Lecture 33: Diagonalization and Applications to Systems of ODEs

Math 3013
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# Diagonalizability

## Definition

An $n \times n$ matrix $\mathbf{A}$ is said to be **diagonalizable** if there is an invertible $n \times n$ matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$$

## Theorem

*An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ has n linearly independent eigenvectors.*

# Constructing the matrices C and D

## Theorem

*Suppose an* $n \times n$ *matrix* $\mathbf{A}$ *has* $n$ *linearly independent eigenvectors* $\mathbf{v}_1, \ldots, \mathbf{v}_n$*. Let* $\lambda_1, \ldots, \lambda_n$ *be the corresponding set of eigenvalues, so that*

$$\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad , \quad i = 1, \ldots, n$$

*From these eigenvector/eigenvalue pairs, construct two matrices*

$$\mathbf{C} \equiv \begin{bmatrix} \uparrow & \cdots & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & \cdots \downarrow & \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} \quad .$$

*Then*

$$\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$$

*In other words,* $\mathbf{C}$ *diagonalizes* $\mathbf{A}$*.*

# Situations $\mathbf{A}$ is automatically diagonalizable

### Theorem

*Suppose $\mathbf{A}$ is an $n \times n$ matrix, and let $p_{\mathbf{A}}(\lambda)$ be its characteristic polynomial*

$$p_{\mathbf{A}}(\lambda) \equiv \det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - r_1)^{m_1} \cdots (\lambda - r_k)^{m_k}$$

*and so its eigenvalues are $r_1, \ldots, r_k$. Then*

- *If $\mathbf{A}$ has $n$ distinct eigenvalues (all $m_i = 1$), then $\mathbf{A}$ is diagonalizable.*
- *If $m_i = \mu_i \equiv \dim(E_{r_i})$ for all $i = 1, \ldots k$, then $\mathbf{A}$ is diagonalizable.*
- *If $\mathbf{A} = \mathbf{A}^t$, then $\mathbf{A}$ is diagonalizable.*

# Example 1.

Determine if the matrix $\mathbf{A} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}$ is diagonalizable; and, if so, find a matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$.

We have

$$\begin{aligned} p_{\mathbf{A}}(\lambda) &= \det(\mathbf{A} - \lambda\mathbf{I}) \\ &= (5-\lambda)(2-\lambda) + 2 \\ &= \lambda^2 - 7\lambda + 12 \\ &= (\lambda - 3)(\lambda - 4) \end{aligned}$$

We thus have two distinct eigenvalues, $\lambda = 3, 4$. And so $\mathbf{A}$ is diagonalizable.
We'll next find the corresponding eigenvectors:

# $\lambda = 3$ Eigenspace

$$
\begin{aligned}
Null\left(\mathbf{A} - (3)\,\mathbf{I}\right) &= NullSp\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \\
&= NullSp\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \\
&= span\left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right) \\
&= span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)
\end{aligned}
$$

So, we can use $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_1 = 3$.

## $\lambda = 4$ Eigenspace

$$
\begin{aligned}
NullSp\,(\mathbf{A} - (4)\,\mathbf{I}) &= NullSp \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \\
&= NullSp \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\
&= span \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)
\end{aligned}
$$

and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda_2 = 4$.

Having found two linearly independent eigenvectors $\mathbf{v}_1$ and

$$\begin{aligned} \mathbf{v}_1 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad , \quad \lambda_1 = 3 \\ \mathbf{v}_2 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad \lambda_2 = 4 \end{aligned}$$

We can use $\mathbf{v}_1$ and $\mathbf{v}_2$ to form a diagonalizing matrix $\mathbf{C}$:

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

The corresponding diagonal matrix $\mathbf{D}$ is then formed by writing the corresponding eigenvalues of these eigenvectors in the same order along its main diagonal:

$$\mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

# Application: Diagonalization and Systems of ODEs

Consider the following $2 \times 2$ system of first order ODEs:

$$\begin{aligned}\frac{dx_1}{dt}(t) &= a_{11}x_1(t) + a_{12}x_2(t) \\ \frac{dx_2}{dt}(t) &= a_{21}x_1(t) + a_{22}x_2(t)\end{aligned}$$

Such systems occur in a number of disparate contexts

- Chemistry. The rate at which the concentration of a reactant changes is proportional to its concentration and the concentration of another reactant.
- Biology. The rate at which a predator and prey populations changes is related to the populations of predators and prey.
- Physics. Coupled oscillators
- Electrical Engineering. Simple passive element (LRC) circuits

# Matrix Formulation of a System of Linear ODEs

Set

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad , \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

so that we can write the system as matrix/differential equation.

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t) \qquad (*)$$

We'll regard (*) as the Linear Algebraic reformulation of the original system.

# Case 1: **A** is a Diagonal Matrix

Suppose

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

In this case, we say that the system is **decoupled**; because the differential equations for such a system are of the form

$$\begin{aligned} \frac{dx_1}{dt} &= \lambda_1 x_1 + 0 \\ \frac{dx_2}{dt} &= 0 + \lambda_2 x_2 \end{aligned}$$

Such equations are easily solved, one-at-a-time,

$$\begin{aligned} x_1(t) &= c_1 e^{\lambda_1 t} \\ x_2(t) &= c_2 e^{\lambda_2 t} \end{aligned}$$

# Case 2: $\mathbf{A}$ is not Diagonal, but is Diagonalizable

This is the general case that we want to solve.

So suppose $\mathbf{A}$ is diagonalizable and that we have found the eigenvalues $\lambda_1, \lambda_2$ and eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of the coefficient matrix $\mathbf{A}$, as well as the matrices $\mathbf{C}$ and $\mathbf{D}$ such that

$$\mathbf{C} = \begin{pmatrix} \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 \\ \downarrow & \downarrow \end{pmatrix} \quad , \quad \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

with

$$\mathbf{D} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C} \quad \iff \quad \mathbf{A} = \mathbf{C}\mathbf{D}\mathbf{C}^{-1}$$

# Diagonalizable Matrix $\mathbf{A}$, Cont'd

Now consider the related system of ODEs corresponding to the diagonal matrix $\mathbf{D}$

$$\frac{d}{dt}\mathbf{y}(t) = \mathbf{D}\mathbf{y}(t)$$

or

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \end{bmatrix}$$

This is a decoupled system which will have

$$\mathbf{y}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

as its general solution

# Diagonalizable Matrix **A**, Cont'd

Now consider

$$\mathbf{x}(t) = \mathbf{C}\mathbf{y}(t)$$

This vector function will satisfy

$$\begin{aligned}
\frac{d}{dt}\mathbf{x}(t) &= \frac{d}{dt}(\mathbf{C}\mathbf{y}(t)) \\
&= \mathbf{C}\frac{d}{dt}\mathbf{y}(t) \qquad \text{since } \mathbf{C} \text{ is a constant matrix} \\
&= \mathbf{C}(\mathbf{D}\mathbf{y}(t)) \\
&= \mathbf{C}\mathbf{D}\mathbf{C}^{-1}\mathbf{C}\mathbf{y}(t) \\
&= \left(\mathbf{C}\mathbf{D}\mathbf{C}^{-1}\right)(\mathbf{C}\mathbf{y}(t)) \\
&= \mathbf{A}\mathbf{x}(t)
\end{aligned}$$

That is to say,

$$\mathbf{x}(t) = \mathbf{C}\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

will satisy the original system of coupled ODEs (in fact, it will be the general solution).

# Summary: Solving Systems of Linear ODEs via Diagonalization

One can solve a system of coupled ODEs

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}x_1(t) + \cdots + a_{1n}x_n(t)\\ &\vdots\\ \frac{dx_n}{dt} &= a_{n1}x_1(t) + \cdots + a_{nn}x_n(t)\end{aligned}$$

by carrying out the following sequence of steps:

# Summary: Solving Systems of Linear ODEs via Diagonalization, Cont'd

1. Form the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

2. Find the eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbf{A}$, and use them to form the diagonal matrix $\mathbf{D}$ and the diagonalizing matrix $\mathbf{C}$

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & & \downarrow \end{bmatrix}$$

# Summary: Solving Systems of Linear ODEs via Diagonalization, Cont'd

3. Solve the decoupled system (easy)

$$\frac{d\mathbf{y}}{dt} = \mathbf{D}\mathbf{y}(t) \quad \Rightarrow \quad \mathbf{y}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

4. Transform the decoupled solutions back to solutions $\mathbf{x}(t)$ of the original system

$$\mathbf{x}(t) = \mathbf{C}\mathbf{y}(t)$$