Lecture 34 : Orthogonal Decompositions

Math 3013
Oklahoma State University

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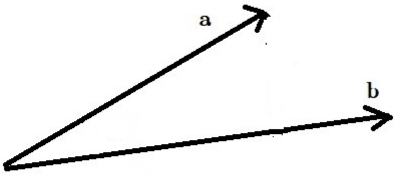
Agenda:

1. Orthogonal Decompositions of Vectors
2. Orthogonal Projection of a Vector onto a Vector
3. Orthogonal Projections onto Subspaces
4. Examples

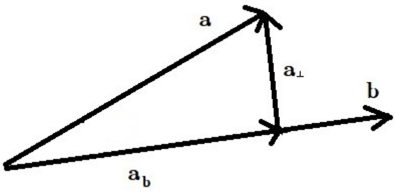
Orthogonal Decompositions of Vectors

Consider two vectors in a plane



Orthogonal Decompositions of Vectors

Consider two vectors in a plane



From the diagram above, we have

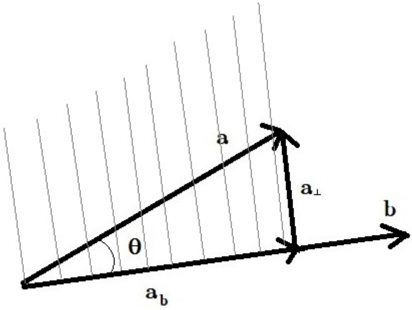
$$\mathbf{a} = \mathbf{a_b} + \mathbf{a}_\perp \qquad (*)$$

We say that

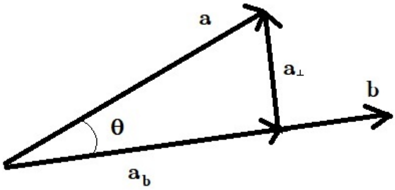
- Equation (*) is the **orthogonal decomposition** of a vector **a** with respect to the direction of **b**,
- $\mathbf{a_b}$ is **the component of a in the direction of b**, and
- $\mathbf{a}_\perp$ is **the component of a perpendicular to b**

Orthogonal Projections

The vector $\mathbf{a_b}$ is also called **the orthogonal projection of a on b**, because if we had a flashlight oriented perpendicularly to the vector **b**, the “shadow” of the vector **a** along **b** would be precisely the vector $\mathbf{a}_b$.



Let's now bring in a little high school trigonometry.



We have

$$\|\mathbf{a_b}\| = \|\mathbf{a}\| \cos(\theta)$$
$$\|\mathbf{a}_\perp\| = \|\mathbf{a}\| \sin(\theta)$$

Back to Linear Algebra

Early on in this course, we learned that the vector dot product

$$\mathbf{a} \cdot \mathbf{b} \equiv a_1 b_1 + \cdots + a_n b_n$$

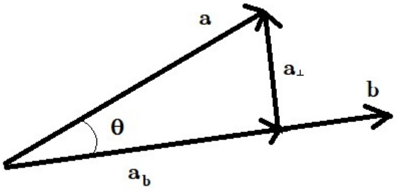
provides us with a means of determining lengths and angles in $\mathbb{R}^n$.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos \theta_{\mathbf{ab}}$$

Here

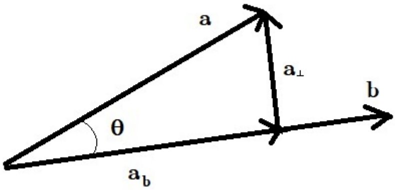
$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{\mathbf{a} \cdot \mathbf{a}} \equiv \text{the length of } \mathbf{a} \\ \|\mathbf{b}\| &= \sqrt{\mathbf{b} \cdot \mathbf{b}} \equiv \text{the length of } \mathbf{b} \\ \theta_{\mathbf{ab}} &= \text{the angle between } \mathbf{a} \text{ and } \mathbf{b} \text{ in the plane spanned by } \mathbf{a} \text{ and } \mathbf{b} \end{aligned}$$

So from



we have

$$\begin{aligned} \|\mathbf{a_b}\| &= \|\mathbf{a}\| \cos\left(\theta_{\mathbf{ab}}\right) \\ &= \|\mathbf{a}\| \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} \\ &= \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|} \end{aligned}$$



Since $\mathbf{a_b}$ has the same direction as $\mathbf{b}$, we can recover the vector $\mathbf{a_b}$ by multiplying the unit vector

$$\widehat{\mathbf{b}} = \frac{\mathbf{b}}{\|\mathbf{b}\|}$$

in the direction of $\mathbf{b}$ by the length $\|\mathbf{a_b}\|$.
And so

$$\mathbf{a_b} = \|\mathbf{a_b}\| \, \widehat{\mathbf{b}} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|}\frac{\mathbf{b}}{\|\mathbf{b}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b}$$

Thus,

Theorem

Let $\mathbf{a}$ *and* $\mathbf{b}$ *be two vectors in* $\mathbb{R}^n$. *Write*

$$\mathbf{a} = \mathbf{a}_{\mathbf{b}} + \mathbf{a}_{\perp}$$

for the orthogonal decomposition of $\mathbf{a}$ *with respect to the direction of* $\mathbf{b}$. *Then*

$$\begin{aligned} \mathbf{a}_{\mathbf{b}} &= \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} \\ \mathbf{a}_{\perp} &= \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} \end{aligned}$$

Example 1

Find the orthogonal decomposition of $\mathbf{a} = [1, 2, 1]$ with respect to the direction of $\mathbf{b} = [1, -1, 0]$.

We have

$$\begin{aligned} \mathbf{a_b} &= \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \mathbf{b} \\ &= \frac{(1)(1) + (2)(-1) + (1)(0)}{(1)^2 + (-1)^2 + (0)^2} [1, -1, 0] \\ &= -\frac{1}{2} [1, -1, 0] = \left[\frac{1}{2}, -\frac{1}{2}, 0 \right] \end{aligned}$$

and

$$\begin{aligned} \mathbf{a}_\perp &= \mathbf{a} - \mathbf{a}_b \\ &= [1, 2, 1] - \left[\frac{1}{2}, -\frac{1}{2}, 0 \right] \\ &= \left[\frac{1}{2}, \frac{5}{2}, 1 \right] \end{aligned}$$

Example 1, Cont'd

and so, the orthogonal decomposition of $\mathbf{a} = [1, 2, 1]$ with respect to the direction of $\mathbf{b} = [1, -1, 0]$ is

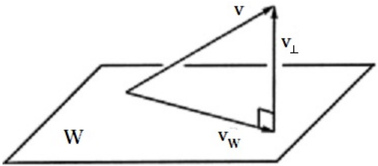
$$\begin{aligned} \mathbf{a} &= \mathbf{a}_{\mathbf{b}} + \mathbf{a}_{\perp} \\ &= \left[\frac{1}{2}, -\frac{1}{2}, 0\right] + \left[\frac{1}{2}, \frac{5}{2}, 1\right] \end{aligned}$$

Othogonal Projections onto Subspaces

We'll now generalize these ideas.

Problem

Given a vector $\mathbf{v} \in \mathbb{R}^n$ *and a subspace* W *of* $\mathbb{R}^n$*. What component of* $\mathbf{v}$ *lies along the directions in* W*?*



Othogonal Projections onto Subspaces, Cont'd

We will show that there are unique vectors $\mathbf{v}_\perp$ and $\mathbf{v}_W$ such that

- $\mathbf{v}_W \in W$
- $\mathbf{v}_\perp$ is perpendicular to every vector in W
- $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$

We will call $\mathbf{v}_W$ the **orthogonal projection** of a vector $\mathbf{v}$ **onto** W. It will be exactly the component of $\mathbf{v}$ that lies in the subspace W.

The equation $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$ is called the **orthogonal decomposition** of the vector $\mathbf{v}$ with respect to the subspace W.

Digression: The Orthogonal Complement, $W^\perp$, of a subspace W

Let W be a k-dimensional subspace with basis $B_W = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$.

The first thing we shall do is construct a subspace $W^\perp$ of $\mathbb{R}^n$ that is perpendicular to every vector in W.
That is to say, we seek a subspace $W^\perp \subset \mathbb{R}^n$ such that

$$\mathbf{v} \in W^\perp \quad \Longrightarrow \quad \mathbf{v} \cdot \mathbf{w} = 0 \qquad \text{for every vector } \mathbf{w} \in W$$

Since every vector in W can be written

$$\mathbf{w} = w_1\mathbf{b}_1 + w_2\mathbf{b}_2 + \cdots + w_k\mathbf{b}_k$$

an easy way to impose the condition $\mathbf{v} \cdot \mathbf{w} = 0$ for all vectors $\mathbf{w} \in W$, would be to demand

$$\mathbf{v} \cdot \mathbf{b}_i = 0 \qquad \text{for } i = 1, \ldots, k$$

These k conditions on $\mathbf{v}$ can then be expressed as a matrix equation

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \cdot \mathbf{v} \\ \vdots \\ \mathbf{b}_k \cdot \mathbf{v} \end{bmatrix} = \begin{bmatrix} \longleftarrow & \mathbf{b}_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \mathbf{b}_k & \longrightarrow \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

In other words, the vector $\mathbf{v}$ will have to lie in the null space of the $k \times n$ matrix formed by using the (n-dimensional) basis vectors $\mathbf{b}_i$ as rows. Set

$$W^{\perp} \equiv NullSp\left(\begin{bmatrix} \longleftarrow & \mathbf{b}_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \mathbf{b}_k & \longrightarrow \end{bmatrix} \right)$$

We have thus set things up so that

$$\mathbf{v} \in W^{\perp} \quad \Longleftrightarrow \quad \mathbf{v} \cdot \mathbf{w} = 0 \quad \text{for all } \mathbf{w} \in W$$

The space $W^{\perp}$ is called the **orthogonal complement to** W **in** $\mathbb{R}^n$

Next, note that since the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_k$ form a basis, they must be linearly independent. Therefore the matrix

$$\mathbf{A}_{W,B} = \begin{bmatrix} \longleftarrow & \mathbf{b}_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \mathbf{b}_k & \longrightarrow \end{bmatrix}$$

has k linearly independent row vectors and so has rank k. But then, since

$$\begin{aligned} n &= \#\text{ columns of } \mathbf{A}_{W,B} \\ &= rank\left(\mathbf{A}_{W,B}\right) + \dim\left(NullSp\left(\mathbf{A}_{W,B}\right)\right) \\ &= k + \dim\left(NullSp\left(\mathbf{A}_{W,B}\right)\right) \end{aligned}$$

So a basis $B_{W^\perp}$ for $W^\perp = NullSp\left(\mathbf{A}_{W,B}\right)$ will have $n-k$ vectors. Let us write such a basis as

$$B_{W^\perp} = \{\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$$

Lemma

The set $\{\mathbf{b}_1, \dots, \mathbf{b}_k, \mathbf{b}_{k+1}, \dots, \mathbf{b}_n\}$ where $\{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is our given basis for W and $\{\mathbf{b}_{k+1}, \dots, \mathbf{b}_n\}$ is a basis for the null space of $\mathbf{A}_{W,B}$, is a basis for $\mathbb{R}^n$.

Proof.

$$c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k + c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n = \mathbf{0}$$

Then we'd have

$$c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k = -c_{k+1}\mathbf{b}_{k+1} - \cdots - c_n\mathbf{b}_n \qquad (*)$$

Set

$$\begin{aligned} \mathbf{v}_1 &= c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k \in W \\ \mathbf{v}_2 &= c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n \in W^{\perp} \end{aligned}$$

so that (*) becomes

$$\mathbf{v}_1 = -\mathbf{v}_2 \qquad (**)$$

Proof of Lemma, Cont'd

Since every vector in $W^{\perp}$ is perpendicular to every vector in W, we must have

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

But then if we take the dot product of both sides of (**) with $\mathbf{v}_1$, we get

$$\|\mathbf{v}_1\|^2 = \mathbf{v}_1 \cdot \mathbf{v}_1 = -\mathbf{v}_1 \cdot \mathbf{v}_2 = 0 \quad \Rightarrow \quad \mathbf{v}_1 = \mathbf{0}$$

But then (**) implies

$$\mathbf{0} = \mathbf{v}_1 = -\mathbf{v}_2 \quad \Rightarrow \quad \mathbf{v}_2 = 0$$

Finally, since $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for W

$$\mathbf{0} = \mathbf{v}_1 = c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k \quad \Rightarrow \quad c_1 = 0, \ldots, c_k = 0$$

and, similarly, since $\{\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ is a basis for $W^{\perp}$, we have

$$\mathbf{0} = \mathbf{v}_2 = c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n \quad \Rightarrow \quad c_{k+1} = 0, \ldots, c_n = 0$$

Thus, all coefficients $c_1, \ldots, c_n$ must separately vanish.

Proof of Lemma, Cont'd

So we've shown that

$$c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k + c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n = \mathbf{0}$$

requires

$$c_1 = 0, \ldots, c_n = 0$$

Thus, the n vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ are linearly independent, and hence form a basis for $\mathbf{R}^n$.

Let's now return to the original problem of finding the orthogonal decomposition of a vector $\mathbf{v}$ with respect to a subspace W.

Theorem

Let W be a subspace of $\mathbb{R}^n$. Then every vector $\mathbf{v}$ in $\mathbb{R}^n$ has a unique decomposition

$$\mathbf{v} = \mathbf{v}_W + \mathbf{v}_{W^\perp}$$

with $\mathbf{v}_W \in W$ and $\mathbf{v}_{W^\perp} \in W^\perp$.

Sketch of Proof. We again fix a basis $B_W = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ of W and a basis $B_{W^\perp} = \{\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ for $W^\perp$ where

$$W^\perp = NullSp\left(\begin{bmatrix} \longleftarrow & \mathbf{b}_1 & \longrightarrow \\ & \vdots & \\ \longleftarrow & \mathbf{b}_k & \longrightarrow \end{bmatrix}\right)$$

The preceding lemma tells us that $B_W \cup B_{W^\perp}$ is a basis for $\mathbb{R}^n$.

Proof of Theorem, Cont'd

Thus, every vector $\mathbf{v} \in \mathbb{R}^n$ has a unique expression as

$$\begin{aligned}
\mathbf{v} &= c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k + c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n \\
&= (c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k) + (c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n) \\
&= \mathbf{v}_W + \mathbf{v}_{W^\perp}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{v}_W &= c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k \in W \\
\mathbf{v}_{W^\perp} &= c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n \in W^\perp
\end{aligned}$$

□

Algorithm for Determining $\mathbf{v}_W$ and $\mathbf{v}_{W^\perp}$

We now summarize the algorithms used in the Lemma and Theorem to obtain the splitting $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_{W^\perp}$.

- Find a basis $B_W = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ for W
- Find a basis $B_{W^\perp} = \{\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ for $W^\perp = \mathit{NullSp}\left(\mathbf{A}_{W,B}\right)$
- Find the coordinate vector $\mathbf{v}_B$ of $\mathbf{v}$ with respect to the basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ of $\mathbb{R}^n$ using the row reduction

$$\left[\begin{array}{ccc|c} | & & | & | \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n & \mathbf{v} \\ | & & | & | \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & \cdots & 0 & | \\ \vdots & \cdots & \vdots & \mathbf{v}_B \\ 0 & & 1 & | \end{array}\right] = [\mathbf{I} \mid \mathbf{v}_B]$$

Algorithm for Determining $\mathbf{v}_W$ and $\mathbf{v}_{W^\perp}$, Cont'd

- Set

$$\begin{aligned} \mathbf{v}_W &= c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k \\ \mathbf{v}_{W^\perp} &= c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n \end{aligned}$$

  where c_i, is the i^{th} component of the coordinate vector $\mathbf{v}_B$.
- We then have

$$\mathbf{v} = \mathbf{v}_W + \mathbf{v}_{W^\perp}$$

  as the orthogonal decomposition of $\mathbf{v}$ with respect to the subspace W.