Lecture 35 : Orthogonal Decompositions: Examples

Math 3013
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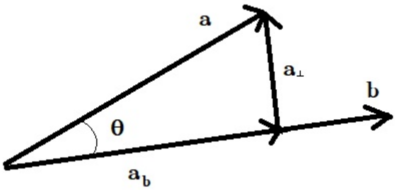
1. Orthogonal Decompositions (Summary)
2. Examples

1: Orthogonal Decompositions: Summary

1. The Orthogonal Decomposition of a Vector **a** with respect to a Vector **b**



$$\mathbf{a} = \mathbf{a_b} + \mathbf{a}_\perp$$

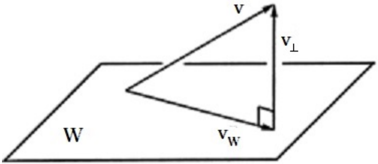
where

$$\mathbf{a_b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} \quad , \quad \text{the component of } \mathbf{a} \text{ in the direction of } \mathbf{b}$$

$$\mathbf{a}_\perp = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} \quad , \quad \text{the component of } \mathbf{a} \text{ perpendicular to } \mathbf{b}$$

Orthogonal Decompositions: Summary Cont'd

2. The Orthogonal Decomposition of a Vector **v** with respect to a subspace $W \subseteq \mathbb{R}^n$



$$\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$$

To identify the components $\mathbf{v}_W$ and $\mathbf{v}_\perp$, we first decompose $\mathbb{R}^n$ into two perpendicular subspaces W and $W_\perp$.

Determinining $\mathbf{v}_W$ and $\mathbf{v}_\perp$

- Find a basis $B_W = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ for W
- Define $W_\perp = \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}$
- Lemma: $W_\perp = NullSp\left(\begin{bmatrix} \leftarrow & \mathbf{b}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{b}_k & \rightarrow \end{bmatrix}\right).$
- Find a basis $B_{W_\perp} = \{\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ for $W_\perp$ by solving the homogeneous linear system

$$\begin{bmatrix} \leftarrow & \mathbf{b}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{b}_k & \rightarrow \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ is then a basis for $\mathbb{R}^n$.

- Find the coordinate vector $\mathbf{v}_B$ of $\mathbf{v} \in \mathbb{R}^n$ with respect to the basis B

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \quad \Longleftrightarrow \quad \mathbf{v}_B = [c_1, \ldots, c_n]$$

by row reducing the corresponding augmented matrix

$$\left[\begin{array}{ccc|c} \uparrow & & \uparrow & \uparrow \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n & \mathbf{v} \\ \downarrow & & \downarrow & \downarrow \end{array}\right] \longrightarrow [\mathbf{I} \mid \mathbf{v}_B]$$

- We then have

$$\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$$

where

$$\begin{aligned} \mathbf{v}_W &= c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k \\ \mathbf{v}_\perp &= c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n \end{aligned}$$

Example 1

Let $W = span([1,0,1],[0,1,1]) \subset \mathbb{R}^3$. Decompose the vector $\mathbf{v} = [1,4,-4]$ into its components $\mathbf{v}_W \in W$ and $\mathbf{v}_{W_\perp} \in W_\perp$.

The two vectors $\mathbf{b}_1 \equiv [1,0,1]$ and $\mathbf{b}_2 \equiv [0,1,1]$ are obviously linearly independent and so $B_W = \{\mathbf{b}_1, \mathbf{b}_2\}$ is already a basis for W. To get a basis for $W_\perp$, we compute the null space of

$$\mathbf{A}_{W,B} = \begin{bmatrix} \leftarrow & \mathbf{b}_1 & \rightarrow \\ \leftarrow & \mathbf{b}_2 & \rightarrow \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

This matrix is already in reduced row echelon form and its null space will be the solution set of

$$\left.\begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\} \quad \Longrightarrow \quad \mathbf{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \quad B_{W_\perp} = \{[-1,-1,1]\} \equiv \{\mathbf{b}_3\}$$

Example 1, Cont'd

We now compute the coordinate vector of $\mathbf{v} = [1, 4, -4]$ with respect to the basis

$$B = B_W \cup B_{W_\perp} = \{[1, 0, 1], [0, 1, 1], [-1, -1, 1]\}$$

of $\mathbb{R}^3$

$$\left[\begin{array}{ccc|c} \uparrow & \cdots & \uparrow & \uparrow \\ \mathbf{b}_1 & \cdots & \mathbf{b}_n & \mathbf{v} \\ \downarrow & \cdots & \downarrow & \downarrow \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 4 \\ 1 & 1 & 1 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

So $\mathbf{v}_B = [-2, 1, -3]$. But now

$$\mathbf{v} = (-2)\,\mathbf{b}_1 + (1)\,\mathbf{b}_2 + (-3)\,\mathbf{b}_3$$

and so

$$\begin{aligned} \mathbf{v}_W &= (-2)\,\mathbf{b}_1 + (1)\,\mathbf{b}_2 = [-2, 1, -1] \\ \mathbf{v}_{W_\perp} &= (-3)\,\mathbf{b}_3 = [3, 3, -3] \end{aligned}$$

Example 2

Find the projection of the vector $\mathbf{v} = [1, 2, 1]$ on the solution set of $x_1 + x_2 + x_3 = 0$.

Let

$$W = \left\{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0 \right\}$$

Solving the linear equation, we find that

$$W = span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

and $\{[-1, 1, 0], [-1, 0, 1]\}$ is a basis for W. $W_\perp$ will then be

$$\begin{aligned} NullSp\left(\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right) &= NullSp\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \right) \\ &= span\left([1, 1, 1] \right) \end{aligned}$$

Example 2, Cont'd

So we have split $\mathbb{R}^n$ into two subspaces

$$W = span\left([-1,1,0],[-1,0,1]\right) \quad , \quad W_{\perp} = span\left([1,1,1]\right)$$

and $B = \{[-1,1,0],[-1,0,1],[1,1,1]\}$ is a basis for $\mathbb{R}^n$.

To identify $\mathbf{v}_W$, the component of the vector $\mathbf{v}$ that lies in W, we need the first two components of the coordinate vector of $\mathbf{v}$ with respect to the basis B.

$$\left[\begin{array}{ccc|c} -1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{array}\right]$$

So

$$\mathbf{v}_W = \frac{2}{3}[-1,1,0] - \frac{1}{3}[-1,0,1] = \left[-\frac{1}{3},\frac{2}{3},\frac{-1}{3}\right]$$

Review: Row Reduction Calculations

Row Echelon Form and Reduced Row Echelon Form

$$R.E.F. \sim \begin{bmatrix} \underline{*} & * & * & * & * & * \\ 0 & 0 & \underline{*} & * & * & * \\ 0 & 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & 0 & \underline{*} & * \end{bmatrix} \quad , \quad R.R.E.F. \sim \begin{bmatrix} \underline{1} & * & 0 & 0 & 0 & * \\ 0 & 0 & \underline{1} & 0 & 0 & * \\ 0 & 0 & 0 & \underline{1} & 0 & * \\ 0 & 0 & 0 & 0 & \underline{1} & * \end{bmatrix}$$

Elementary Row Operations

$$\begin{array}{ccc} R_i & \longleftrightarrow & R_j \\ R_i & \to & \lambda R_i \\ R_i & \to & R_i + \lambda R_j \end{array}$$

Applications of Row Reduction

(1) Solving Linear Systems $\mathbf{Ax} = \mathbf{b}$: row reduce $[\mathbf{A}|\mathbf{b}]$ to its R.R.E.F.
Important Special Case:

$$NullSp(\mathbf{A}) = \text{ solution set of } \mathbf{Ax} = \mathbf{0}$$

(2) Finding basis for subspace $W = span(\mathbf{v}_1, \ldots, \mathbf{v}_k)$: row reduce $\begin{bmatrix} \leftarrow & \mathbf{v}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{v}_k & \rightarrow \end{bmatrix}$ to R.E.F. and grab non-zero rows

Applications of Row Reduction, Cont'd

(3) Finding bases for subspaces attached to matrices: If $\mathbf{A}'' = R.R.E.F.(\mathbf{A})$

(i) basis for $RowSp(\mathbf{A}) \longleftrightarrow$ nonzero rows of $\mathbf{A}''$

(ii) basis for $ColSp(\mathbf{A}) \longleftrightarrow$ columns of $\mathbf{A}$ corresponding to columns of $\mathbf{A}''$ that contain pivots

(iii) basis for $NullSp(\mathbf{A}) \longleftrightarrow$ constant vectors that occur in hyperplane expression for the solution set of $\mathbf{Ax} = \mathbf{0}$

(4) Finding $\mathbf{A}^{-1}$: $R.R.E.F.([\mathbf{A}|\mathbf{I}]) = \left[\mathbf{I}|\mathbf{A}^{-1}\right]$ (if $\mathbf{A}^{-1}$ exists)

Applications of Row Reduction, Cont'd

(5) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation

$$
\begin{aligned}
\mathbf{A}_T &= \begin{bmatrix} \uparrow & & \uparrow \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_m) \\ \downarrow & & \downarrow \end{bmatrix} \\
Range(T) &= ColSp(\mathbf{A}_T) \\
Ker(T) &= NullSp(\mathbf{A}_T)
\end{aligned}
$$

Applications of Row Reduction, Cont'd

(6) Calculating Determinants: Suppose $\mathbf{A} \to \mathbf{A}' = R.E.F.(\mathbf{A})$

$$\det(\mathbf{A}) = (-1)^i \frac{1}{\lambda_1 \cdots \lambda_k} a'_{11} a'_{22} \cdots a'_{nn}$$

where

$$\begin{aligned} i &= \#\text{ of row interchanges } R_i \longleftrightarrow R_j \text{ used} \\ \lambda_i &= \text{scalar factors used via } R_i \to \lambda R_i \text{ type operations} \\ a'_{ii} &= \text{diagonal entries of the R.E.F. } \mathbf{A}' \text{ of } \mathbf{A} \end{aligned}$$

Applications of Row Reduction, Cont'd

(7) Finding eigenvectors (i.e., basis vectors for eigenspaces)

$$E_r = NullSp\,(\mathbf{A} - r\mathbf{I})$$