Lecture 38 : General Vector Spaces

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Lecture 37

Agenda:

1. Generalized Vector Spaces
2. Definitions and Theorems for Generalized Vector Spaces
3. Bases and Coordinatization of Generalized Vector Spaces
4. Example

Generalized Vector Spaces

Notation. Let S and T be two sets. The **Cartesian product** of S and T, denoted $S \times T$, is the set

$$S \times T = \{(s, t) \mid s \in S \text{ and } t \in T\}$$

i.e., the set of all ordered pairs of elements of S and T.

Definition

A vector space over $\mathbb{R}$ is a set V for which the following operations are defined

- **scalar multiplication**: for every $\lambda \in \mathbb{R}$ and $\mathbf{v} \in V$ we have a map $*_V : \mathbb{R} \times V \to V : (\lambda, \mathbf{v}) \to \lambda *_V \mathbf{v} \in V$.
- **vector addition:** for every pair of vectors $\mathbf{u}, \mathbf{v} \in V$ we have a map $+_V : V \times V \to V : (\mathbf{u}, \mathbf{v}) \to \mathbf{u} +_V \mathbf{v} \in V$

Definition of Generalized Vector Spaces, Cont'd

In addition, the operations of vector addition and scalar multiplication must also satisfy the following axioms

1. $(\mathbf{u} +_V \mathbf{v}) +_V \mathbf{w} = \mathbf{u} +_V (\mathbf{v} +_V \mathbf{w})$ (associativity of vector addition)
2. $\mathbf{u} +_V \mathbf{v} = \mathbf{v} +_V \mathbf{u}$ (commutativity of vector addition)
3. There exists an element $\mathbf{0}_V \in V$ such that $\mathbf{v} +_V \mathbf{0}_V = \mathbf{v}$ for all $\mathbf{v} \in V$. (additive identity.)
4. For each $\mathbf{v} \in V$ there exists an element $-\mathbf{v} \in V$ such that $\mathbf{v} +_{\mathbf{V}} (-\mathbf{v}) = \mathbf{0}_V$. (additive inverses)
5. $\lambda *_V (\mathbf{u} +_V \mathbf{v}) = \lambda *_V \mathbf{u} +_V \lambda *_V \mathbf{v}$ (scalar multiplication is distributive with respect to vector addition).
6. $(\lambda +_{\mathbb{R}} \mu) *_v \mathbf{v} = \lambda *_V \mathbf{v} +_V \mu *_V \mathbf{v}$. (scalar multiplication is distributes over addition of scalars)
7. $\lambda *_V (\mu *_V \mathbf{v}) = (\lambda *_{\mathbb{R}} \mu)\mathbf{v}$ (scalar multiplication preserves associativity of multiplication in $\mathbb{R}$.)
8. $(1) *_V \mathbf{v} = \mathbf{v}$ (preservation of scale).

Example of a General Vector Space

Definition. Let $V \equiv \{f : \mathbb{R} \to \mathbb{R}\}$ be the set of all functions on the real line together with the following two operations

- Scalar Multiplication:

$$*_V : \mathbb{R} \times V \to V \quad , \quad (\lambda, v) \to \lambda *_V v$$

where $\lambda *_V v$ is the function

$$(\lambda *_V v)(x) \equiv \lambda *_{\mathbb{R}} v(x)$$

- Vector Addition

$$+_V : V \times V \to V \quad , \quad (f, g) \to f +_V g$$

where

$$(f +_V g)(x) \equiv f(x) +_{\mathbb{R}} g(x)$$

Note: We regard a function $f : \mathbb{R} \to \mathbb{R}$ as being uniquely identified once we provide a formula for the values $f(x)$ at all points $x \in \mathbb{R}$.

Example : $V = \{f : \mathbb{R} \to \mathbb{R}\}$

Theorem

The set V together with its operations $*_V$ and $+_V$ is a general vector space over $\mathbb{R}$.

Proof. We have 8 axioms to verify.

$V = \{f : \mathbb{R} \to \mathbb{R}\}$: 1. Associativity of Vector Addition

Let $f, g, h \in V$. We need to show

$$(f +_V g) +_V h = f +_V (g +_V h) \tag{1}$$

Evaluating the left hand side at a point $x \in \mathbb{R}$, we have

$$\begin{aligned} [(f +_V g) +_V h](x) &\equiv (f +_V g)(x) +_{\mathbb{R}} h(x) \\ &= (f(x) +_{\mathbb{R}} g(x)) +_{\mathbb{R}} h(x) \end{aligned}$$

while, on the right, we'd have

$$\begin{aligned} [f +_V (g +_V h)](x) &\equiv f(x) +_{\mathbb{R}} (g +_V h)(x) \\ &= f(x) +_{\mathbb{R}} (g(x) +_{\mathbb{R}} h(x)) \end{aligned}$$

Since addition of real numbers is associative, we have

$$(f(x) +_{\mathbb{R}} g(x)) +_{\mathbb{R}} h(x) = f(x) +_{\mathbb{R}} (g(x) +_{\mathbb{R}} h(x)) \quad , \quad \forall\, x \in \mathbb{R}$$

$V = \{f : \mathbb{R} \to \mathbb{R}\}$: 1. Associativity of Vector Addition, Cont'd

And so

$$[(f +_V g) +_V h]\,(x) = [f +_V (g +_V h)]\,(x) \quad , \quad \forall\, x \in \mathbb{R}$$

Since the value of the function $(f +_V g) +_V h$ is exactly the same as the value of the function $f +_V (g +_V h)$ at all points $x \in \mathbb{R}$, the two functions must be same and so (1) is proved.

$V = \{f : \mathbb{R} \to \mathbb{R}\}$: 2. Commutativity of Vector Addition

Let $f, g \in V$. We need to show

$$f +_V g = g +_V f \tag{2}$$

To show that the two functions are equal, we evaluate both sides at an arbitrary point $x \in \mathbb{R}$.

$$\begin{array}{ccc} (f +_V g)(x) & \stackrel{?}{=} & (g +_V f)(x) \\ \shortparallel & & \shortparallel \\ f(x) +_{\mathbb{R}} g(x) & = & g(x) +_{\mathbb{R}} f(x) \\ & \checkmark & \end{array}$$

because addition of real numbers is commutative.

$V = \{f : \mathbb{R} \to \mathbb{R}\}$: 3. Additive Identity

Let $f_{\mathbf{0}}$ be the function defined by

$$f_{\mathbf{0}}(x) = 0 \quad , \quad \forall\, x \in \mathbb{R}$$

Then $f_{\mathbf{0}} \in V$, and, for any other function $g \in V$

$$(f_{\mathbf{0}} +_V g) = g \tag{3}$$

For

$$(f_{\mathbf{0}} +_V g)(x) = f_{\mathbf{0}}(x) +_{\mathbb{R}} g(x) = 0 +_{\mathbb{R}} g(x) = g(x) \quad , \quad \forall\, x \in \mathbb{R}$$

And so $f_{\mathbf{0}}$ acts as the zero vector $\mathbf{0}_V$ in V.

$V = \{f : \mathbb{R} \to \mathbb{R}\}$: 4. Additive Inverses

For any $f \in V$, let $-f$ be the function defined by

$$(-f)(x) \equiv (-1) *_{\mathbb{R}} f(x)$$

Note $-f \in V$ and we have

$$(f +_V (-f))(x) = f(x) - f(x) = 0 \quad , \quad \forall \; x \in \mathbb{R}$$

and so

$$f + (-f) = f_{\mathbf{0}} \equiv \mathbf{0} \in V \quad , \quad \forall \; f \in V \tag{4}$$

Example $\{f : \mathbb{R} \to \mathbb{R}\}$ - Conclusion

We have four more properties to prove.

5. $\lambda *_V (\mathbf{u} +_V \mathbf{v}) = \lambda *_V \mathbf{u} +_V \lambda *_V \mathbf{v}$
6. $(\lambda +_{\mathbb{R}} \mu) *_v \mathbf{v} = \lambda *_V \mathbf{v} +_V \mu *_V \mathbf{v}$.
7. $\lambda *_V (\mu *_V \mathbf{v}) = (\lambda *_{\mathbb{R}} \mu)\mathbf{v}$
8. $(1) *_V \mathbf{v} = \mathbf{v}$

However, just as for the first two axioms, these properties follow by **meticulously** applying the definitions of $*_V$ and $+_V$ and then using a corresponding property of real numbers.
I'll forgo the proofs of the last four properties/axioms. □

Definitions and Theorems for Generalized Vector Spaces

Henceforth, we'll simplify our notation, using $+$ for $+_V$, and $\lambda\mathbf{v}$ for $\lambda *_V \mathbf{v}$.

- **Def.** Let V be a generalized vector space and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of vectors in V: A **linear combination** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an expression of the form

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \qquad , \qquad c_1, \ldots, c_k \in \mathbb{R}$$

- **Def.** The **span** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the set of all possible linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$:

$$\mathit{span}\left(\mathbf{v}_1, \ldots, \mathbf{v}_k\right) = \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}$$

Definitions and Theorems, Cont'd

- **Def.** A **subspace** of a vector space V is a subset W of V such that

  (i) $\lambda \in \mathbb{R}, \mathbf{w} \in W \implies \lambda\mathbf{w} \in W$

  (ii) $\mathbf{w}_1, \mathbf{w}_2 \in W \implies \mathbf{w}_1 + \mathbf{w}_2 \in W$

  With this notion of subspace, and the following theorem, we can now produce many, many examples of general vector spaces

- **Thm.** Let W be a subspace of a generalized vector space V. Define operations $*_W : \mathbb{R} \times W \to W$ and $+_W : W \times W \to W$ by

$$\begin{aligned} \lambda *_W \mathbf{w} &\equiv \lambda *_V \mathbf{w} \quad \forall \mathbf{w} \in W \\ \mathbf{w}_1 +_W \mathbf{w}_2 &\equiv \mathbf{w}_1 +_V \mathbf{w}_2 \quad \forall \mathbf{w}_1, \mathbf{w}_2 \in W \end{aligned}$$

  Then W together with the operations $*_W$ and $+_W$ is a general vector space.

Definitions and Theorems, Cont'd

- **Def.** A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is **linearly independent** if the only solution of

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

  is the trivial solution where $x_1 = 0, x_2 = 0, \ldots, x_k = 0$.
- **Def.** A **basis** for a general vector space V is a set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subset V$ such that
  - Every $\mathbf{v} \in V$ can be expressed as

$$\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

  - The vectors $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ are linearly independent.
- **Thm.** Every general vector space V has a basis and every basis for a general vector space V has the same number of vectors.
- **Def.** The **dimension** of a general vector space V is the number of vectors in any basis for V.

Definitions and Theorems, Cont'd

- Def. A **linear transformation** is a function $T : V \to W$ between two vector spaces such that
  - $T(\lambda \mathbf{v}) = \lambda \mathbf{v}$ for all $\lambda \in \mathbb{R}$ and for all $\mathbf{v} \in V$
  - $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in V$
- Def. The **range** of a linear transformation $T : V \to W$ is the set

$$range(T) = \{\mathbf{w} \in W \mid \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}$$

- Def. The **kernel** of a linear transformation $T : V \to W$ is the set

$$\ker(T) = \{\mathbf{v} \in V \mid T(\mathbf{v}) = \mathbf{0}_W\}$$

- Def. A **vector space isomorphism** is an invertible linear transformation $T : V \to W$ between two vector spaces.
- **Theorem:** $T : V \to W$ is a vector space isomorphism if and only if $Range(T) = W$ and $\ker(T) = \{\mathbf{0}_V\}$

Coordinatization of Generalized Vector Spaces

The idea of bases is absolutely vital for dealing with generalized vector spaces. They provide us with coordinates by which we can carry out numerical calculations.

This works essentially the same way we used a basis for a subspace W of $\mathbb{R}^n$ to provide good coordinates for the vectors in W.

Definition

Let $\mathbf{v}$ be a vector in a generalized vector space V and let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V. The **coordinate vector** for $\mathbf{v}$ with respect to B is the unique element of $\mathbb{R}^n$ defined by

$$\mathbf{v}_B = [c_1, c_2, \ldots, c_n]$$

where the numbers $c_1, c_2, \ldots, c_n$ are determined by the unique expansion

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

Coordinatization of Generalized Vector Spaces, Cont'd

Bases thus allow us to attach to "abstract" vectors $\mathbf{v} \in V$ concrete numerical coordinate vectors $\mathbf{v}_B \in \mathbb{R}^n$.

In fact,

Theorem

The correspondence $V \ni \mathbf{v} \mapsto \mathbf{v}_B \in \mathbb{R}^n$ *defines an isomorphism (i.e., an invertible linear transformation)* $i_B : V \to \mathbb{R}^n$

Corollary

Every finite-dimensional vector space is isomorphic to a particular $\mathbb{R}^n$*.*

Calculational Procedure for Generalized Vector Spaces

$$\begin{array}{ccc} \text{problem in } V & \xrightarrow{i_B} & \text{problem in } \mathbb{R}^n \\ & & \downarrow \text{ numerical calculations} \\ \text{solution in } V & \xleftarrow{i_B^{-1}} & \text{solution in } \mathbb{R}^n \end{array}$$

N.B. This procedure first requires a basis for V.

Example: Solving a Differential Equation using Linear Algebra

Consider

$$x^2 \frac{d^2 f}{dx^2} - 4x \frac{df}{dx} + 6f = 0$$

We shall look for solutions of this differential equation inside the vector space $\mathcal{P}_4$ of polynomials of degree ≤ 4.

The first thing we'll need is a basis for $\mathcal{P}_4$. Luckily, there's a natural one. Every polynomial of degree ≤ 4 is an expression the form $a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$. In fact,

$$\begin{aligned} \mathcal{P}_4 &= \left\{ a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \mid a_4, \ldots, a_0 \in \mathbb{R} \right\} \\ &= span\left(1, x, x^2, x^3, x^4\right) \end{aligned}$$

Moreover, it's pretty obvious that $\left\{1, x, x^2, x^3, x^4\right\}$ are linearly independent polynomials since

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \quad \Rightarrow \quad a_0 = 0 \,,\; a_1 = 0 \,,\; \ldots \,,\; a_4 = 0$$

So $B = \left\{1, x, x^2, x^3, x^4\right\}$ is a set of linearly independent vectors that generate $\mathcal{P}_4$ - hence, B is a basis for the vector space $\mathcal{P}_4$.

We then have the following coordinatization isomorphism

$$i_B : \mathcal{P}_4 \to \mathbb{R}^5 \quad : \quad a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \longleftrightarrow [a_0, a_1, a_2, a_3, a_4]$$

which will allow us to convert the original problem about polynomials in $\mathcal{P}_4$ to a problem in $\mathbb{R}^5$.

Solving $x^2\frac{d^2f}{dx^2} - 4x\frac{df}{dx} + 6f = 0$, Cont'd

Now consider the differential operator on the left hand side of the differential equation

$$\mathcal{L} \equiv x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6$$

We have

$$\begin{aligned} \mathcal{L}(\lambda f) &= x^2\frac{d^2}{dx^2}(\lambda f) - 4x\frac{d}{dx}(\lambda f) + 6(\lambda f) \\ &= \lambda x^2\frac{d^2f}{dx^2} - 4\lambda x\frac{df}{dx} + 6\lambda f \\ &= \lambda\mathcal{L}(f) \end{aligned}$$

$$\begin{aligned} \mathcal{L}(f+g) &= x^2\frac{d^2}{dx^2}(f+g) - 4x\frac{d}{dx}(f+g) + 6(f+g) \\ &= x^2\frac{d^2f}{dx^2} - 4x\frac{df}{dx} + 6f + x^2\frac{d^2g}{dx^2} - 4x\frac{dg}{dx} + 6g \\ &= \mathcal{L}(f) + \mathcal{L}(g) \end{aligned}$$

Example, Cont'd

Thus, $\mathcal{L} : \mathcal{P}_4 \to \mathcal{P}_4$ is a linear transformation.

Moreover, solving the original differential equation is equivalent to finding the kernel of $\mathcal{L}$:

$$\ker(\mathcal{L}) = \left\{ f \in \mathcal{P}_4 \mid x^2 \frac{d^2 f}{dx^2} - 4x \frac{df}{dx} + 6f = 0 \right\}$$

So the question now is how to calculate $\ker(\mathcal{L})$?

N.B. We have now reformulated the problem of solving a differential equation to an equivalent linear algebraic problem in $\mathbf{P}_4$.

Calculating $\ker(\mathcal{L})$

Recall that to find the kernel of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$, we construct a matrix

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ T([1,0,0,0,0]) & \cdots & T([0,0,0,0,1]) \\ \downarrow & \cdots & \downarrow \end{bmatrix}$$

and then calculate $NullSp(\mathbf{A}_T) = \{\mathbf{x} \in \mathbb{R}^5 \mid \mathbf{A}_T\mathbf{x} = \mathbf{0}\}$

We'll now use the coordinate isomorphism $i_B : \mathcal{P}_4 \to \mathbb{R}^5$ and its inverse $i_B^{-1} : \mathbb{R}^5 \to \mathcal{P}_4$ to convert the problem of finding $\ker(\mathcal{L}) \subseteq \mathcal{P}_4$ to a problem in $\mathbb{R}^5$.

Let's first look at how $\mathcal{L}$ acts on the basis vectors for $\mathcal{P}_4$:

$$
\begin{aligned}
\mathcal{L}(1) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)(1) = 0 + 0 + 6 = 6 \\
\mathcal{L}(x) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)(x) = 0 - 4x + 6x = 2x \\
\mathcal{L}\left(x^2\right) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)\left(x^2\right) = 2x^2 - 8x^2 + 6x^2 = 0 \\
\mathcal{L}\left(x^3\right) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)\left(x^3\right) = 6x^3 - 12x^2 + 6x^3 = 0 \\
\mathcal{L}\left(x^4\right) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)\left(x^4\right) = 12x^4 - 16x^4 + 6x^4 = 2x^4
\end{aligned}
$$

We thus have

$$\begin{aligned}
\mathcal{L}(1) &= 6(1) \\
\mathcal{L}(x) &= 2(x) \\
\mathcal{L}\left(x^2\right) &= 0 \\
\mathcal{L}\left(x^3\right) &= 0 \\
\mathcal{L}\left(x^4\right) &= 2\left(x^4\right)
\end{aligned}$$

Let's now employ the coordinatization isomorphism $i_B : \mathcal{P}_4 \leftrightarrow \mathbb{R}^5$.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^5 \longleftrightarrow [a_0, a_1, a_2, a_3, a_4]$$

$$\begin{array}{ccr} 1 & \xrightarrow{\mathcal{L}} & 6\,(1) \\ x & \xrightarrow{\mathcal{L}} & 2\,(x) \\ x^2 & \xrightarrow{\mathcal{L}} & 0 \\ x^3 & \xrightarrow{\mathcal{L}} & 0 \\ x^4 & \xrightarrow{\mathcal{L}} & 2x^4 \end{array}$$

Let's now employ the coordinatization isomorphism $i_B : \mathcal{P}_4 \leftrightarrow \mathbb{R}^5$.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^5 \longleftrightarrow [a_0, a_1, a_2, a_3, a_4]$$

$$\begin{array}{ccccccc}
[1,0,0,0,0] & \xrightarrow{i_B^{-1}} & 1 & \xrightarrow{\mathcal{L}} & 6\,(1) & \xrightarrow{i_B} & [6,0,0,0,0] \\
[0,1,0,0,0] & \xrightarrow{i_B^{-1}} & x & \xrightarrow{\mathcal{L}} & 2\,(x) & \xrightarrow{i_B} & [0,2,0,0,0] \\
[0,0,1,0,0] & \xrightarrow{i_B^{-1}} & x^2 & \xrightarrow{\mathcal{L}} & 0 & \xrightarrow{i_B} & [0,0,0,0,0] \\
[0,0,0,1,0] & \xrightarrow{i_B^{-1}} & x^3 & \xrightarrow{\mathcal{L}} & 0 & \xrightarrow{i_B} & [0,0,0,0,0] \\
[0,0,0,0,1] & \xrightarrow{i_B^{-1}} & x^4 & \xrightarrow{\mathcal{L}} & 2\left(x^4\right) & \xrightarrow{i_B} & [0,0,0,0,2]
\end{array}$$

which shows how the linear transformation $i_B \circ \mathcal{L} \circ i_B^{-1} : \mathbb{R}^5 \to \mathbb{R}^5$ acts on the standard basis vectors of $\mathbb{R}^5$.

If we now define

$$T : \mathbb{R}^5 \to \mathbb{R}^5 : \mathbf{x} \longmapsto i_B \circ \mathcal{L} \circ i_B^{-1}$$

then

$$\mathbf{A}_T = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

We can now solve the related problem in $\mathbb{R}^5$:

Solving $\mathcal{L}(p) = 0$ in $\mathcal{P}_4 \longleftrightarrow$ Finding $Ker(T) = NullSp\,(\mathbf{A}_T)$ in $\mathbb{R}^5$

After row reducing $\mathbf{A}_T$, one sees

$$R.R.E.F.(\mathbf{A}_T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the solutions of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ must have

$$x_1 = 0 \quad , \quad x_2 = 0 \quad , \quad x_5 = 0$$

and x_3 and x_4 are left as free parameters.

Thus, a solution vector must be of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and the constant vectors on the right hand side will be the basis vectors for $NullSp\left(\mathbf{A}_T\right)$.
Hence,

$$\ker\left(T\right) = NullSp\left(\mathbf{A}_T\right) = span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

This is our "answer" in $\mathbb{R}^5$.

However, the original problem was posed in $\mathcal{P}_4$. To find the appropriate answer in $\mathcal{P}_4$, we need to use $i_B^{-1} : \mathbb{R}^5 \to \mathcal{P}_4$ to pull our answer in $\mathbb{R}^5$ back to $\mathcal{P}_4$.

$$
\begin{aligned}
\ker(\mathcal{L}) &= i_B^{-1} \ker(A_T) \\
&= i_B^{-1}\left(span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right)\right) \\
&= span\left(x^2, x^3\right) \\
&= \left\{ c_1 x^2 + c_2 x^3 \mid c_1, c_2 \in \mathbb{R} \right\}
\end{aligned}
$$

□