Lecture 39 : Calculating in a General Vector Space

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Lecture 38 : Calculating in a General Vector Space

Agenda:

1. Review: Bases and Coordinatization
2. Example: Solving a Linear ODE using Linear Algebra

Coordinatization of Generalized Vector Spaces

Definition

Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a general vector space V and let $\mathbf{v} \in V$. Then $\mathbf{v}$ has a unique expression as a linear combination of the vectors in B

$$\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

The **coordinate vector** of $\mathbf{v}$ with respect to the basis B is the element

$$\mathbf{v}_B = [c_1, c_2, \dots, c_n] \in \mathbb{R}^n.$$

Theorem

The map $i_B : V \to \mathbb{R}^n : \mathbf{v} \mapsto \mathbf{v}_B$ is an isomorphism of vector spaces.

Calculational Scheme for Generalized Vector Spaces

$$\begin{array}{ccc}
\text{problem in } V & \xrightarrow{\quad i_B \quad} & \text{problem in } \mathbb{R}^n \\
 & & \downarrow \text{ numerical calculations} \\
\text{solution in } V & \xleftarrow{\quad i_B^{-1} \quad} & \text{solution in } \mathbb{R}^n
\end{array}$$

But note: **This procedure first requires a basis B for V.**

Example: Solving a Differential Equation using Linear Algebra

Consider

$$x^2 \frac{d^2 f}{dx^2} - 4x \frac{df}{dx} + 6f = 0$$

We shall look for solutions of this differential equation inside the vector space $\mathcal{P}_4$ of polynomials of degree ≤ 4.

The first thing we'll need is a basis for $\mathcal{P}_4$.
Now, a polynomial of degree ≤ 4 is an expression the form $a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$. In fact,

$$\begin{aligned} \mathcal{P}_4 &= \left\{ a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \mid a_4, \ldots, a_0 \in \mathbb{R} \right\} \\ &= span\left(1, x, x^2, x^3, x^4\right) \end{aligned}$$

So, as a vector space, $\mathcal{P}_4$ is generated by the monomials $1, x, x^2, x^3, x^4$.

Moreover, it's pretty obvious that $\left\{1, x, x^2, x^3, x^4\right\}$ are linearly independent polynomials
For

$$a_4x^4+a_3x^3+a_2x^2+a_1x+a_0=0 \quad \Rightarrow \quad a_0=0\,,\ a_1=0\,,\ \ldots\ ,\ a_4=0$$

So $B=\left\{1, x, x^2, x^3, x^4\right\}$ is a set of linearly independent vectors that generate $\mathcal{P}_4$ - hence, B is a basis for the vector space $\mathcal{P}_4$.

We then have the following coordinatization isomorphism

$$i_B : \mathcal{P}_4 \leftrightarrow \mathbb{R}^5 \ :\ a_4x^4+a_3x^3+a_2x^2+a_1x+a_0 \longleftrightarrow [a_0, a_1, a_2, a_3, a_4]$$

which will allow us to convert the original problem about polynomials in $\mathcal{P}_4$ to a problem in $\mathbb{R}^5$.

Solving $x^2 \frac{d^2 f}{dx^2} - 4x \frac{df}{dx} + 6f = 0$, Cont'd

Now consider the differential operator on the left hand side of the differential equation

$$\mathcal{L} \equiv x^2 \frac{d^2}{dx^2} - 4x \frac{d}{dx} + 6$$

We have $\mathcal{L} : \mathcal{P}_4 \to \mathcal{P}_4$

$$\begin{aligned} \mathcal{L}(\lambda f) &= x^2 \frac{d^2}{dx^2}(\lambda f) - 4x \frac{d}{dx}(\lambda f) + 6(\lambda f) \\ &= \lambda x^2 \frac{d^2 f}{dx^2} - 4\lambda x \frac{df}{dx} + 6\lambda f \\ &= \lambda \mathcal{L}(f) \end{aligned}$$

$$\begin{aligned} \mathcal{L}(f+g) &= x^2 \frac{d^2}{dx^2}(f+g) - 4x \frac{d}{dx}(f+g) + 6(f+g) \\ &= x^2 \frac{d^2 f}{dx^2} - 4x \frac{df}{dx} + 6f + x^2 \frac{d^2 g}{dx^2} - 4x \frac{dg}{dx} + 6g \\ &= \mathcal{L}(f) + \mathcal{L}(g) \end{aligned}$$

Example, Cont'd

Thus,

$$\begin{aligned}\mathcal{L}(\lambda f) &= \lambda \mathcal{L}(f) \\ \mathcal{L}(f+g) &= \mathcal{L}(f) + \mathcal{L}(g)\end{aligned}$$

and so $\mathcal{L} : \mathcal{P}_4 \to \mathcal{P}_4$ is a linear transformation of the vector space $\mathbf{P}_4$.

Moreover, solving the original differential equation is equivalent to finding the kernel of $\mathcal{L}$:

$$\ker(\mathcal{L}) = \left\{ f \in \mathcal{P}_4 \mid x^2 \frac{d^2 f}{dx^2} - 4x \frac{df}{dx} + 6f = 0 \right\}$$

We have now reformulated the problem of solving a differential equation to an equivalent linear algebraic problem in the vector space $\mathcal{P}_4$.
So the question now is how to calculate $\ker(\mathcal{L})$?

Calculating the Kernel of a Linear Transformation

Recall that to find the kernel of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$, we construct a matrix

$$\mathbf{A}_T = \left[\begin{array}{ccc} \uparrow & \cdots & \uparrow \\ T([1,0,0,0,0]) & \cdots & T([0,0,0,0,1]) \\ \downarrow & \cdots & \downarrow \end{array} \right]$$

and then calculate $\mathit{NullSp}(\mathbf{A}_T) = \left\{ \mathbf{x} \in \mathbb{R}^5 \mid \mathbf{A}_T \mathbf{x} = \mathbf{0} \right\}$

We'll now use the coordinate isomorphism $i_B : \mathcal{P}_4 \to \mathbb{R}^5$ and its inverse $i_B^{-1} : \mathbb{R}^5 \to \mathcal{P}_4$ to convert the problem of finding $\ker(\mathcal{L}) \subseteq \mathcal{P}_4$ to a problem in $\mathbb{R}^5$.

Let's first look at how $\mathcal{L}$ acts on the basis vectors for $\mathcal{P}_4$:

$$
\begin{aligned}
\mathcal{L}(1) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)(1) = 0 + 0 + 6 = 6 \\
\mathcal{L}(x) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)(x) = 0 - 4x + 6x = 2x \\
\mathcal{L}\left(x^2\right) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)\left(x^2\right) = 2x^2 - 8x^2 + 6x^2 = 0 \\
\mathcal{L}\left(x^3\right) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)\left(x^3\right) = 6x^3 - 12x^2 + 6x^3 = 0 \\
\mathcal{L}\left(x^4\right) &= \left(x^2\frac{d^2}{dx^2} - 4x\frac{d}{dx} + 6\right)\left(x^4\right) = 12x^4 - 16x^4 + 6x^4 = 2x^4
\end{aligned}
$$

Let's re-express these results in terms of the basis $B = \left\{1, x, x^2, x^3, x^4\right\}$ of $\mathcal{P}_4$:

We thus have

$$\begin{aligned}
\mathcal{L}(1) &= 6(1) \\
\mathcal{L}(x) &= 2(x) \\
\mathcal{L}\left(x^2\right) &= 0 \\
\mathcal{L}\left(x^3\right) &= 0 \\
\mathcal{L}\left(x^4\right) &= 2\left(x^4\right)
\end{aligned}$$

Let's now employ the coordinatization isomorphism $i_B : \mathcal{P}_4 \leftrightarrow \mathbb{R}^5$.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^5 \longleftrightarrow [a_0, a_1, a_2, a_3, a_4]$$

$$\begin{array}{lcl} 1 & \xrightarrow{\mathcal{L}} & 6\,(1) \\ x & \xrightarrow{\mathcal{L}} & 2\,(x) \\ x^2 & \xrightarrow{\mathcal{L}} & 0 \\ x^3 & \xrightarrow{\mathcal{L}} & 0 \\ x^4 & \xrightarrow{\mathcal{L}} & 2x^4 \end{array}$$

Let's now employ the coordinatization isomorphism $i_B : \mathcal{P}_4 \leftrightarrow \mathbb{R}^5$.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^5 \longleftrightarrow [a_0, a_1, a_2, a_3, a_4]$$

$$\begin{array}{ccccccc}
[1,0,0,0,0] & \xrightarrow{i_B^{-1}} & 1 & \xrightarrow{\mathcal{L}} & 6\,(1) & \xrightarrow{i_B} & [6,0,0,0,0] \\
 & & x & \xrightarrow{\mathcal{L}} & 2\,(x) & & \\
 & & x^2 & \xrightarrow{\mathcal{L}} & 0 & & \\
 & & x^3 & \xrightarrow{\mathcal{L}} & 0 & & \\
 & & x^4 & \xrightarrow{\mathcal{L}} & 2x^4 & &
\end{array}$$

Let's now employ the coordinatization isomorphism $i_B : \mathcal{P}_4 \leftrightarrow \mathbb{R}^5$.

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^5 \longleftrightarrow [a_0, a_1, a_2, a_3, a_4]$$

$$\begin{array}{ccccccc}
[1,0,0,0,0] & \xrightarrow{i_B^{-1}} & 1 & \xrightarrow{\mathcal{L}} & 6\,(1) & \xrightarrow{i_B} & [6,0,0,0,0] \\
[0,1,0,0,0] & \xrightarrow{i_B^{-1}} & x & \xrightarrow{\mathcal{L}} & 2\,(x) & \xrightarrow{i_B} & [0,2,0,0,0] \\
[0,0,1,0,0] & \xrightarrow{i_B^{-1}} & x^2 & \xrightarrow{\mathcal{L}} & 0 & \xrightarrow{i_B} & [0,0,0,0,0] \\
[0,0,0,1,0] & \xrightarrow{i_B^{-1}} & x^3 & \xrightarrow{\mathcal{L}} & 0 & \xrightarrow{i_B} & [0,0,0,0,0] \\
[0,0,0,0,1] & \xrightarrow{i_B^{-1}} & x^4 & \xrightarrow{\mathcal{L}} & 2\left(x^4\right) & \xrightarrow{i_B} & [0,0,0,0,2]
\end{array}$$

which shows how the linear transformation $i_B \circ \mathcal{L} \circ i_B^{-1} : \mathbb{R}^5 \to \mathbb{R}^5$ acts on the standard basis vectors of $\mathbb{R}^5$.

If we now define

$$T : \mathbb{R}^5 \to \mathbb{R}^5 : \ \mathbf{x} \mapsto i_B \circ \mathcal{L} \circ i_B^{-1}(\mathbf{x})$$

then

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_5) \\ \downarrow & \cdots & \downarrow \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

We can now solve the related problem in $\mathbb{R}^5$:

$$\text{Solving } \mathcal{L}(f) = 0 \text{ in } \mathcal{P}_4 \longleftrightarrow \text{ Finding } Ker(T) \in \mathbb{R}^5$$

where

$$Ker(T) = NullSp(\mathbf{A}_T) \subset \mathbb{R}^5$$

Determining $NullSp(\mathbf{A}_T)$

After row reducing $\mathbf{A}_T$, one sees

$$R.R.E.F.(\mathbf{A}_T) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the solutions of $\mathbf{A}_T\mathbf{x} = \mathbf{0}$ must have

$$x_1 = 0 \quad , \quad x_2 = 0 \quad , \quad x_5 = 0$$

and x_3 and x_4 are left as free parameters.

Thus, a solution vector must be of the form

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and the constant vectors on the right hand side will be the basis vectors for $NullSp\left(\mathbf{A}_T\right)$.
Hence,

$$\ker\left(T\right) = NullSp\left(\mathbf{A}_T\right) = span\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

This is our "answer in $\mathbb{R}^5$" .

However, the original problem was posed in $\mathcal{P}_4$. To find the appropriate "answer in $\mathcal{P}_4$", we need to use $i_B^{-1} : \mathbb{R}^5 \to \mathcal{P}_4$ to pull our answer in $\mathbb{R}^5$ back to $\mathcal{P}_4$.

$$
\begin{aligned}
\ker(\mathcal{L}) &= i_B^{-1} \ker(A_T) \\
&= i_B^{-1} \left(span \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) \right) \\
&= span \left(x^2, x^3 \right) \\
&= \left\{ c_1 x^2 + c_2 x^3 \mid c_1, c_2 \in \mathbb{R} \right\}
\end{aligned}
$$

Thus the solution set of the original differential equation

$$\mathcal{L}(f) = x^2 \frac{d^2 f}{dx^2} - 4x \frac{df}{dx} + 6f = 0$$

is

$$Ker(\mathcal{L}) = \left\{ c_1 x^2 + c_2 x^3 \mid c_1, c_2 \in \mathbb{R} \right\}$$

$\square$