Lecture 40 : Inner Products for General Vector Spaces

Math 3013
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Lecture 39: Inner Products for Vector Spaces

Agenda

1. Review of Orthonormal Bases
2. Inner Products for General Vector Spaces
3. Application: Fourier Analysis

Review of Orthonormal Bases

Definition

A set of vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an **orthonormal** basis for a subspace W of $\mathbb{R}^n$ if

(i) $W = span\,(\mathbf{w}_1, \ldots, \mathbf{w}_k)$;

(ii) The vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ are linearly independent;

(iii) $\mathbf{w}_i \cdot \mathbf{w}_j = 0$ if $i \neq j$; (orthogonality)

(iv) $\mathbf{w}_i \cdot \mathbf{w}_i = 1$ (the vectors have unit length)

Suppose $B = \{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is an orthonormal basis for a subspace W.
Then by (i), every $\mathbf{v} \in W$ can be expressed as

$$\mathbf{v} = c_1\mathbf{w}_1 + \cdots + c_k\mathbf{w}_k \qquad (*)$$

(ii) ensures that the coefficients $c_1, \ldots, c_k$ on the right hand sides of (*) are unique (and so provide good coordinates for $\mathbf{v}$).

Orthonormal Bases Cont'd

By conditions (iii) and (iv), the values of the coefficients c_i are easily determined
For

$$\begin{aligned} \mathbf{w}_i \cdot \mathbf{v} &= \mathbf{w}_i \cdot (c_1 \mathbf{w}_1 + \cdots + c_i \mathbf{w}_i + \cdots + c_k \mathbf{w}_k) \\ &= c_1 \mathbf{w}_i \cdot \mathbf{w}_1 + \cdots + c_i \mathbf{w}_i \cdot \mathbf{w}_i + \cdots + c_k \mathbf{w}_i \cdot \mathbf{w}_k \\ &= 0 + \cdots + 0 + c_i (1) + 0 + \cdots + 0 \\ &= c_i \end{aligned}$$

and so

$$c_i = \mathbf{w}_i \cdot \mathbf{v}$$

Today's Topic

Last week we described an algorithm, the so-called Gram-Schmidt Process, that constructs an orthonormal basis for a subspace W from any other basis for W.

Today, we will generalize this apparatus to more general vector spaces.

The first thing we'll need is a generalization of the dot product (which in $\mathbb{R}^n$ is used to test for orthogonality).

Inner Products for Vector Spaces

Here is our generalization of the dot product for $\mathbb{R}^n$:

Definition

Let V be a (general) vector space over $\mathbb{R}$. An **inner product** on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

with the following properties:

(i) Linearity

$$\begin{aligned} \langle \lambda v, w \rangle &= \lambda \langle v, w \rangle = \langle v, \lambda w \rangle \qquad \forall\ \lambda \in \mathbb{R} \\ \langle v_1 + v_2, w \rangle &= \langle v_1, w \rangle + \langle v_2, w \rangle \end{aligned}$$

(ii) Symmetry

$$\langle v, w \rangle = \langle w, v \rangle$$

(iii) Positive Definiteness

$$\langle v, v \rangle > 0 \quad \text{if } v \neq \mathbf{0}_V$$

Example 1: The dot product on $\mathbb{R}^n$

This first example shows that inner products in general vector spaces are a generalization of the usual dot product in $\mathbb{R}^n$.
Let $\mathbf{v} = [v_1, \ldots, v_n]$, $\mathbf{w} = [w_1, \ldots, w_n]$ and define

$$\langle \mathbf{v}, \mathbf{w} \rangle \equiv \mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

I'll now show that conditions (i), (ii) and (iii) of the definition of an inner product are satisfied.

(i) Linearity

$$\begin{aligned}
\langle \lambda \mathbf{v}, \mathbf{w} \rangle &= (\lambda \mathbf{v}) \cdot \mathbf{w} \\
&= [\lambda v_1, \ldots, \lambda v_n] \cdot [w_1, \ldots, w_n] \\
&= \lambda v_1 w_1 + \cdots + \lambda v_n w_n \\
&= \lambda \left(v_1 w_1 + \cdots + v_n w_n \right) \\
&= \lambda \mathbf{v} \cdot \mathbf{w} \\
&= \lambda \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

Example 1: the dot product on $\mathbb{R}^n$ Cont'd

(i) Linearity Cont'd

$$\begin{aligned}
\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle &= (\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} \\
&= [u_1 + v_1, \ldots, u_n + v_n] \cdot [w_1, \ldots, w_n] \\
&= (u_1 + v_1)\, w_1 + \cdots + (u_n + v_n)\, w_n \\
&= (u_1 w_1 + \cdots + u_n w_n) + (v_1 w_1 + \cdots + v_n w_n) \\
&= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \\
&= \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

Example 1: the dot product on $\mathbb{R}^n$ Cont'd

(ii) Symmetry

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w} \rangle &= \mathbf{v} \cdot \mathbf{w} \\ &= v_1 w_1 + \cdots + v_n w_n \\ &= w_1 v_1 + \cdots + w_n v_n \\ &= \mathbf{w} \cdot \mathbf{v} \\ &= \langle \mathbf{w}, \mathbf{v} \rangle \end{aligned}$$

(iii) Positive-Definiteness

$$\langle \mathbf{v}, \mathbf{v} \rangle = v_1^2 + \cdots + v_n^2 \geq 0$$

since $\langle \mathbf{v}, \mathbf{v} \rangle$ is a sum of squares. In fact, so long as $\mathbf{v} \neq [0, \ldots, 0]$, we have $\langle \mathbf{v}, \mathbf{v} \rangle > 0$.

Thus, the dot product is an inner product for $\mathbb{R}^n$.

Example 2: An Inner Product on a Vector Space of Functions

Let V be the vector space of continuous functions on the interval $[0, L]$ with scalar multiplication and vector addition defined by

$$\begin{aligned} (\lambda f)(x) &\equiv \lambda f(x) \\ (f+g)(x) &\equiv f(x)+g(x) \end{aligned}$$

Define $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ by

$$\langle f, g \rangle \equiv \int_0^L f(x) g(x)\, dx \quad \forall f, g \in V$$

Theorem

With the setup above, $\langle \cdot, \cdot \rangle$ is an inner product on V.

Proof

We need to verify the three properties of an inner product.

(i) Linearity

$$\begin{aligned}
\langle \lambda f, g \rangle &= \int_0^L (\lambda f)(x)\, g(x)\, dx \\
&= \int_0^L \lambda f(x)\, g(x)\, dx \\
&= \lambda \int_0^L f(x)\, g(x)\, dx \\
&= \lambda \langle f, g \rangle
\end{aligned}$$

Example 2: An Inner Product on a Vector Space of Functions, Cont'd

(i) Linearity, Cont'd

$$\begin{aligned} \langle f+g, h\rangle &= \int_0^L (f+g)(x)\, h(x)\, dx \\ &= \int_0^L (f(x)+g(x))\, h(x)\, dx \\ &= \int_0^L f(x)\, h(x)\, dx + \int_0^L g(x)\, h(x)\, dx \\ &= \langle f, h\rangle + \langle g, h\rangle \end{aligned}$$

Example 2: An Inner Product on a Vector Space of Functions, Cont'd

- Symmetry

$$\begin{aligned}
\langle f, g \rangle &= \int_0^L f(x)\, g(x)\, dx \\
&= \int_0^L g(x)\, f(x)\, dx \\
&= \langle g, f \rangle
\end{aligned}$$

Example 2: An Inner Product on a Vector Space of Functions, Cont'd

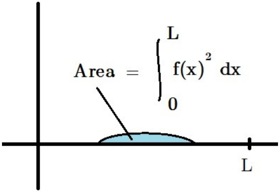
(iii) Positive-Definiteness

$$f \neq 0 \ \Rightarrow^{?} \ \langle f, f \rangle = \int_0^L f(x)^2 \, dx > 0$$

To prove that $\langle f, f \rangle > 0$ for any non-zero function f requires some technical theorems from Calculus.

However, it's not that hard to see, informally, why this must be the case.
One first notes that the integral of a positive function $g(x)$ from 0 to L is just the area under the graph of $g(x)$ between 0 and L.

Inner Product on a Function Space; Positive-Definiteness Cont'd



So long as $f(x)^2$ is non-zero at some point in $[0, L]$, the continuity of $g(x)$ will ensure that there will be some non-zero area under the graph of $f(x)^2$.
And so if $f \neq f_0 = \mathbf{0}_V$

$$\begin{aligned} \langle f, f \rangle &= \int_0^L f(x)^2 \, dx \\ &= \Big(\text{area under the graph of } f(x)^2 \text{ between 0 and } L\Big) \\ &> 0 \end{aligned}$$

since non-zero areas are always positive numbers.

Fourier Series and Harmonic Analysis

Next, consider the functions

$$\phi_n(x) \equiv \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) \ , \ n = 1, 2, 3, \ldots$$

These functions can be thought of as the vibrational modes of string of length L. We have (by direct computation)

$$\langle \phi_n, \phi_m \rangle = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Note how this formula is analogous to the formula

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for the standard basis vectors of $\mathbb{R}^n$.

Fourier Series and Harmonic Analysis, Cont'd

Indeed, the functions $\{\phi_n(x) \mid n \in \mathbb{N}\}$ provide an **orthonormal basis** for V and every function $f \in V$ has a unique expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right) \qquad (*)$$

Equation (*) is called the **Fourier Sine series** for $f(x)$.
It is interpretable as the decomposition of a "vibration" f into its fundamental vibrational modes (a.k.a. its "harmonics").

The numbers b_n are interpretable as the amplitude of the n^{th} vibration mode.

Example 2: Fourier Series and Harmonic Analysis, Cont'd

If we multiply both sides of (*) by $\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$ and integrate over the interval $[0, L]$, we find

$$
\begin{aligned}
\langle \phi_n, f \rangle &= \sqrt{\frac{2}{L}} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx \\
&= \int_0^L \left(\frac{2}{L} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) \right) dx \\
&= \frac{2}{L} \sum_{m=1}^{\infty} \int_0^L b_m \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx \\
&= \sum_{m=1}^{\infty} b_m \delta_{n,m} \\
&= b_n
\end{aligned}
$$

Fourier Series and Harmonic Analysis: Summary

Thus, the coefficients b_n in the harmonic expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) = \sum_{n=1}^{\infty} b_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L} x\right)$$

of f with respect to the orthonormal basis $\{\phi_1, \phi_2, \ldots\}$ are effectively determined by taking the inner product of f with the basis element $\phi_n(x)$.

$$b_n = \langle \phi_n, f \rangle = \sqrt{\frac{2}{L}} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$