Lecture 41 : Review for Final Exam

Math 3013
Oklahoma State University

January 31, 2022

Lecture 40 : Review for Final Exam

Final Exam :

- Section 62663 (MWF, 10:30pm): Friday, December 10, 10:00am - 11:50am
- Section 62667 (MWF, 1:30pm): Wednesday, December 8, 10:00am - 11:50am

See posts on the Math 3013 Canvas homepage for solutions to exams (both midterm exams and sample exams).

Review for Final, Part I.

1. **Vectors and Matrix Algebra**

- Linear Combinations of Vectors, Hyperplanes, Spans

$$\begin{aligned}\mathcal{H} &= \{\mathbf{p}_0 + c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\} \\ span\left(\mathbf{v}_1, \ldots, \mathbf{v}_k\right) &= \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots, c_k \in \mathbb{R}\}\end{aligned}$$

- Matrix Multiplication

$$\left(\mathbf{AB}\right)_{ij} = Row_i\left(\mathbf{A}\right) \cdot Col_j\left(\mathbf{B}\right)$$

2. **Row Reduction**

- Elementary Row Operations $\begin{cases} \mathcal{R}_i \longleftrightarrow \mathcal{R}_j \\ \mathcal{R}_i \to \lambda\mathcal{R}_i \\ \mathcal{R}_i \to \mathcal{R}_i + \lambda\mathcal{R}_j \end{cases}$
- Row Echelon Form and Reduced Row Echelon Form

$$REF \sim \begin{bmatrix} \underline{*} & * & * & * & * \\ 0 & 0 & \underline{*} & * & * \\ 0 & 0 & 0 & 0 & \underline{*} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad , \quad RREF \sim \begin{bmatrix} \underline{1} & * & 0 & * & 0 \\ 0 & 0 & \underline{1} & * & 0 \\ 0 & 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Review for Final, Cont'd

3. Solving Linear Systems $\mathbf{Ax} = \mathbf{b}$

- Augmented Matrices and Hyperplane Solutions

$$
\begin{array}{ccc}
[\mathbf{A}|\mathbf{b}] & \to & RREF\left([\mathbf{A}|\mathbf{b}]\right) \\
\left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] & \to & \begin{cases} x_1 = 2 - x_2 + x_4 \\ x_3 = 3 - x_4 \\ 0 = 0 \end{cases} \\
\mathbf{x} & = & \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}
\end{array}
$$

- Homogeneous Linear Systems and NullSpaces:
  $RREF\left([\mathbf{A}|\mathbf{0}]\right) = \left[RREF\left(\mathbf{A}\right)|\mathbf{0}\right]$

Review for Final, Cont'd

4. Matrix Inverses

- Computing Matrix Inverses via Row Reduction

$$RREF\left([\mathbf{A}|\mathbf{I}]\right) = \left[\mathbf{I}|\mathbf{A}^{-1}\right]$$

$$\nearrow$$

$$[\mathbf{A}|\mathbf{I}]$$

$$\searrow$$

$\mathbf{A}$ has no inverse

Review for Final - Part II

5. Definitions To Be Memorized

- A **subspace** of a vector space V is a subset W of V such that
  - (i) $\lambda \in \mathbb{R}\,,\ \mathbf{w} \in W \Longrightarrow \lambda \mathbf{w} \in W$
  - (ii) $\mathbf{w}_1, \mathbf{w}_2 \in W \Longrightarrow \mathbf{w}_1 + \mathbf{w}_2 \in W$
- A **basis** for a subspace W is a set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ such that
  - (i) every vector $\mathbf{w} \in W$ can be expressed as

$$\mathbf{w} = c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k \qquad (*)$$

  - (ii) The coefficients $c_1, \ldots, c_k$ in (*) are unique.
- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is **linearly independent** if the only solution of

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}$$

  is $x_1 = 0, \ldots, x_k = 0$.
- A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a **linear transformation** if
  - (i) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.
  - (ii) $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^m$

Review for Final Exam, Cont'd

6. Finding Bases for $RowSp(\mathbf{A})$, $ColSp(\mathbf{A})$, $NullSp(\mathbf{A})$

- basis for $RowSp(\mathbf{A}) \sim$ nonzero rows of $RREF(\mathbf{A})$
- basis for $ColSp(\mathbf{A}) \sim$ columns of $(A|$ corresponding to the columns of $RREF(\mathbf{A})$ with pivots
- basis for $NullSp(\mathbf{A}) \sim$ constant vectors appearing in the hyperplane expression for the solution set of $\mathbf{Ax} = \mathbf{0}$

7. Linear Transformations $T : \mathbb{R}^m \to \mathbb{R}^n$ and Matrices

$$
\begin{aligned}
\mathbf{A}_T &= \begin{bmatrix} \uparrow & \cdots & \uparrow \\ T(\mathbf{e}_1) & \cdots & T(\mathbf{e}_m) \\ \downarrow & \cdots & \downarrow \end{bmatrix} \\
Range(T) &= \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = T(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^m\} \\
&= ColSp(\mathbf{A}_T) \\
Ker(T) &= \{\mathbf{x} \in \mathbb{R}^m \mid T(\mathbf{x}) = \mathbf{0}_{\mathbb{R}^n}\} \\
&= NullSp(\mathbf{A}_T)
\end{aligned}
$$

Review for Final Exam, Cont'd

8. Determinants

- computing determinants via cofactor expansions

$$\det(\mathbf{A}) = \sum a_{ij}(-1)^{i+j}\det(\mathbf{M}_{ij}) \quad , \quad \mathbf{M}_{ij} = (ij)^{th} \text{ minor of } \mathbf{A}$$

- computing determinants via row reduction

$$\det(\mathbf{A}) = (-1)^r \frac{1}{\lambda_1 \cdots \lambda_k} a'_{11} \cdots a'_{nn}$$

where $r = \#$row interchanges used, $\lambda_1, \ldots, \lambda_k$ are the row rescalings used, a'_{ii} the diagonal entries of $\mathbf{A}' \equiv REF(\mathbf{A})$

9. Crammer's Rule Solution of $n \times n$ linear system $\mathbf{Ax} = \mathbf{b}$ is given by

$$x_i = \frac{\det(\mathbf{B}_i)}{\det(\mathbf{A})} \quad , \quad \mathbf{B}_i = \begin{bmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \cdots & \vdots & \cdots & \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}$$

Review for Final Exam, Cont'd

10. Computing $\mathbf{A}^{-1}$ via cofactors

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T \quad , \quad c_{ij} = (-1)^{i+j}\det(\mathbf{M}_{ij})$$

Review for Final : Material covered since the 2nd exam

- Eigenvectors and Eigenvalues
  - Eigenvalues $\leftrightarrow$ solutions of $det\,(\mathbf{A} - \lambda\mathbf{I}) = 0$
  - Eigenvectors with Eigenvalue λ $\leftrightarrow$ solutions of $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$
- Algebraic and Geometric Multiplicities
- Diagonalization of Square Matrices
- Orthogonal Decomposition of a Vector w.r.t. a Subspace
- Gram-Schmidt Process (for constructing an orthonormal basis)

To be reviewed in Wednesday's lecture