Lecture 42 : Review for Final Exam, Part III

Math 3013
Oklahoma State University

January 31, 2022

Lecture 41 : Review for Final Exam, Part III

Final Exam :

- Section 62663 (MWF, 10:30pm): Friday, December 10, 10:00am - 11:50am
- Section 62667 (MWF, 1:30pm): Wednesday, December 8, 10:00am - 11:50am

See posts on the Math 3013 Canvas homepage for solutions to exams (both midterm exams and sample exams).

Review for Final - Part III (Material covered since the 2nd exam)

11. Eigenvectors and Eigenvalues

The **Eigenvector/Eigenvalue Problem** for an $n \times n$ matrix $\mathbf{A}$ is the problem of finding non-zero vectors (eigenvectors) $\mathbf{v}$ and numbers (eigenvalues) λ such that

$$\mathbf{Av} = \lambda \mathbf{v}$$

- the **eigenvalues** of $\mathbf{A}$ are the solutions of $det\left(\mathbf{A} - \lambda \mathbf{I}\right) = 0$
- The **eigenvectors with eigenvalue** λ are solutions of $\left(\mathbf{A} - \lambda \mathbf{I}\right)\mathbf{x} = \mathbf{0}$
- the **algebraic multiplicity** of an eigenvalue r : the number of factors of $(\lambda - r)$ in the characteristic polynomial $p_{\mathbf{A}}(\lambda) \equiv \det\left(\mathbf{A} - \lambda \mathbf{I}\right)$
- **geometric multiplicity** of an eigenvalue r : the dimension of the r-eigenspace $= dim\left(NullSp\left(\mathbf{A} - r\mathbf{I}\right)\right)$

Review for Final - Part III, Cont'd

12. Diagonalization of Square Matrices

Def. An $n \times n$ matrix $\mathbf{A}$ is **diagonalizable** if there exists an invertible $n \times n$ matrix $\mathbf{C}$ and a diagonal $n \times n$ matrix $\mathbf{D}$ such that

$$\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D}$$

- An $n \times n$ matrix $\mathbf{A}$ is diagonalizable **if and only if** it has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, in which case

$$\mathbf{C} = \begin{bmatrix} \uparrow & \cdots & \uparrow \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ \downarrow & \cdots & \downarrow \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

  where λ_i is the eigenvalue corresponding to the eigenvector $\mathbf{v}_i$.
- If an $n \times n$ matrix $\mathbf{A}$ has n distinct eigenvalues then it diagonalizable.
- An $n \times n$ matrix $\mathbf{A}$ is diagonalizable **if and only if** the algebraic multiplicity of each eigenvalue r coincides with its geometric multiplicity.
- If $\mathbf{A}^T = \mathbf{A}$, then $\mathbf{A}$ is diagonalizable.

Review for Final - Part III, Cont'd

13. Orthogonal Decomposition of a Vector w.r.t. a Subspace

- Let W be a subspace of $\mathbb{R}^n$, the **orthogonal complement** of W is the subspace

$$W_\perp \equiv \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{w} \in W\}$$

- Let $\mathbf{v} \in \mathbb{R}^n$ and let W be a subspace of $\mathbb{R}^n$. The **orthogonal decomposition** of $\mathbf{v}$ w.r.t. W is the unique splitting

$$\mathbf{v} = \mathbf{v}_W + \mathbf{v}_\perp$$

where $\mathbf{v}_W \in W$ and $\mathbf{v}_\perp \in W_\perp$.

Review for Final - Part III, Cont'd

- If $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for W, and

  $$\mathbf{B} = \begin{bmatrix} \leftarrow & \mathbf{b}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{b}_k & \rightarrow \end{bmatrix}$$

  then

  $$W_\perp = NullSp(\mathbf{B})$$

- If $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for W and $\{\mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ is a basis for $W_\perp$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$.
  If $\mathbf{v} \in \mathbb{R}^n$, then

  $$\begin{aligned} \mathbf{v} &= c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k + c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n \\ &= (c_1\mathbf{b}_1 + \cdots + c_k\mathbf{b}_k) + (c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n) \\ &= \mathbf{v}_W + \mathbf{v}_\perp \end{aligned}$$

Review for Final - Part III, Cont'd

To get the coefficients $c_1, \ldots, c_k, c_{k+1}, \ldots, c_n$, one calculates the coordinate vector of $\mathbf{v}$ w.r.t. the basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{b}_{k+1}, \ldots, \mathbf{b}_n\}$

$$\left[\begin{array}{cccccc|c} \uparrow & & \uparrow & \uparrow & & & \\ \mathbf{b}_1 & \cdots & \mathbf{b}_k & \mathbf{b}_{k+1} & \cdots & \mathbf{b}_n & \mathbf{v} \\ \downarrow & & \downarrow & \downarrow & & & \end{array}\right]$$

$\downarrow$ row reduction

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & c_1 \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & c_n \end{array}\right]$$

Review for Final - Part III, Cont'd

14. Orthonormal Bases

An orthonormal basis for a subspace W is a basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ such that

$$\mathbf{b}_i \cdot \mathbf{b}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is an orthonormal basis for W then not only can every vector $\mathbf{w} \in W$ be expressed as

$$\mathbf{w} = c_1\mathbf{b}_1 + \cdots + c_k\mathbf{v}_k$$

but also the each coefficient c_i can be determined by

$$c_i = \mathbf{b}_i \cdot \mathbf{w}$$

15. The Gram-Schmidt Process (for constructing an orthonormal basis)

Suppose $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is a basis for a subspace W. Then an orthogonal basis $\{\mathbf{o}_1, \ldots, \mathbf{o}_k\}$ for W can be constructed via the following algorithm:

$$
\begin{aligned}
\mathbf{o}_1 &= \mathbf{b}_1 \\
\mathbf{o}_2 &= \mathbf{b}_2 - \frac{\mathbf{o}_1 \cdot \mathbf{b}_2}{\mathbf{o}_1 \cdot \mathbf{o}_1}\mathbf{o}_1 \\
&\vdots \\
\mathbf{o}_k &= \mathbf{b}_k - \frac{\mathbf{o}_1 \cdot \mathbf{b}_k}{\mathbf{o}_1 \cdot \mathbf{o}_1}\mathbf{o}_1 - \cdots - \frac{\mathbf{o}_{k-1} \cdot \mathbf{b}_k}{\mathbf{o}_{k-1} \cdot \mathbf{o}_{k-1}}\mathbf{o}_{k-1}
\end{aligned}
$$

Review for Final - Part III, Cont'd

Once an orthogonal basis $\{\mathbf{o}_1, \ldots, \mathbf{o}_k\}$ has been constructed, an **orthonormal basis** $\{\mathbf{n}_1, \ldots, \mathbf{n}_k\}$ for W can be constructed by simply rescaling the orthogonal basis vectors so that they have unit length:

$$\begin{aligned}
\mathbf{n}_1 &= \frac{\mathbf{o}_1}{\|\mathbf{o}_1\|} = \frac{\mathbf{o}_1}{\sqrt{\mathbf{o}_1 \cdot \mathbf{o}_1}} \\
\mathbf{n}_2 &= \frac{\mathbf{o}_2}{\|\mathbf{o}_2\|} = \frac{\mathbf{o}_2}{\sqrt{\mathbf{o}_2 \cdot \mathbf{o}_2}} \\
&\vdots \\
\mathbf{n}_k &= \frac{\mathbf{o}_k}{\|\mathbf{o}_k\|} = \frac{\mathbf{o}_k}{\sqrt{\mathbf{o}_k \cdot \mathbf{o}_k}}
\end{aligned}$$