Math 3013
WebAssign Problems #2

1. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, find $\mathbf{u} \cdot \mathbf{v}$.

- $$\mathbf{u} \cdot \mathbf{v} = (-1)(4) + (2)(1) = -2$$

2 . Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, find $\mathbf{u} \cdot \mathbf{v}$.

- $$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (2)(2) + (3)(1) = 10$$

3. Let $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and find $\|\mathbf{u}\|$.

- $$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{(-1)(-1) + (4)(4)} = \sqrt{17}$$

4. Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$

(a) Find $\|\mathbf{u}\|$

- $$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{(3)(3) + (2)(2) + (4)(4)} = \sqrt{29}$$

(b) Give a unit vector in the direction of $\mathbf{u}$.

- $$\hat{\mathbf{u}} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{29}} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{29}} \\ \frac{2}{\sqrt{29}} \\ \frac{4}{\sqrt{29}} \end{bmatrix}$$

5. Find the distance $d(\mathbf{u}, \mathbf{v})$ between $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

- $$\begin{aligned} d(\mathbf{u},\mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \left\| \begin{bmatrix} -1-1 \\ 3-2 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\| \\ &= \sqrt{(-2)(-2)+(1)(1)} \\ &= \sqrt{5} \end{aligned}$$

6. Find the distance $d(\mathbf{u},\mathbf{v})$ between $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$.

- $$\begin{aligned} d(\mathbf{u},\mathbf{v}) &= \|\mathbf{u}-\mathbf{v}\| \\ &= \left\| \begin{bmatrix} 1-2 \\ 2-1 \\ 3-3 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\| \\ &= \sqrt{(-1)(-1)+(1)(1)+(0)(0)} \\ &= \sqrt{2} \end{aligned}$$

7. Let $\mathbf{p} = (1,0)$ and $\mathbf{d} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(a) Write the equation of the line passing through $\mathbf{p}$ in the direction $\mathbf{d}$.

- The desired line is the set of points
$$\begin{aligned} \ell &= \{\mathbf{p}+t\mathbf{d} \mid t \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \end{bmatrix} \mid t \in \mathbb{R} \right\} \end{aligned}$$
So the x and y coordinates will be
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(b) Write the parametric form of the line passing through $\mathbf{p}$ in the direction $\mathbf{d}$.

- To get the parametric form, we simply write down the equations for the components x and y
$$\begin{aligned} x &= 1-t \\ y &= 0+2t \end{aligned}$$

8. Let $\mathbf{p} = (3,0,-2)$ and $\mathbf{d} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

(a) Write the equation of the line passing through $\mathbf{p}$ in the direction $\mathbf{d}$ in vector form.

- We have

$$\begin{aligned} \ell &= \{\mathbf{p} + t\mathbf{d} \mid t \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \right\} \end{aligned}$$

so a typical point on the line will be

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

(b) Write the equation of the line passing through $\mathbf{p}$ in the direction $\mathbf{d}$ in parametric form.

- To get the parametric form, we simply write down the equations for the components x, y, and z

$$\begin{aligned} x &= 3 + 2t \\ y &= 5t \\ z &= -2 \end{aligned}$$

9. Let $\mathbf{p} = (0, 1, 0)$ and $\mathbf{n} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$.

(a) Write the equation of the plane passing through $\mathbf{p}$ with normal vector $\mathbf{n}$ in normal form.

- The vector $\mathbf{n}$ must be perpendicular to each direction in the plane. If $\mathbf{x} = (x, y, z)$ is a point in the plane, $\mathbf{x} - \mathbf{p} = (x, y - 1, z)$ will be a direction lying in the plane and so must be perpendicular to $\mathbf{n}$. Thus

$$0 = \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

(b) Write the equation of the plane passing through $\mathbf{p}$ with normal vector $\mathbf{n}$ in normal form.

- Carrying out the dot product in part (a)

$$0 = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y - 1 \\ z \end{bmatrix} \right) = 4x + 8y - 8 + z$$

or

$$4x + 8y + z = 8$$

10. Let $\mathbf{p} = (0, 0, 0)$, $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$

(a) Write the equation of the plane passing through $\mathbf{p}$ with direction vectors $\mathbf{u}$ and $\mathbf{v}$ in vector form.

- The plane will be the following set of vectors

$$\begin{aligned} P &= \{\mathbf{p} + s\mathbf{v} + t\mathbf{v} \mid s, t \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} + t \begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \right\} \end{aligned}$$

and so a typical vector in the plane will be of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} -4 \\ 5 \\ 1 \end{bmatrix}$$

(b) Write the equation of the plane passing through $\mathbf{p}$ with direction vectors $\mathbf{u}$ and $\mathbf{v}$ in parametric form.

- To tet the parametric form we simply write down the equations for the components

$$\begin{aligned} x &= 5s - 4t \\ y &= s + 5t \\ z &= 5s + t \end{aligned}$$

11. Let $\mathbf{p} = (0, 1, -1)$ and $\mathbf{q} = (-3, 1, 4)$. Give the vector equation of the line passing through $\mathbf{p}$ and $\mathbf{q}$.

- The direction $\mathbf{d}$ of the line will be given by difference of any two points on the line, so, in particular, we can take

$$\mathbf{d} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$$

And the line through $\mathbf{p}$ in the direction $\mathbf{d}$ will be

$$\ell = \{\mathbf{p} + t\mathbf{d} \mid t \in \mathbb{R}\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

so a typical vector on the line will be of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$$

12.

(a) Consider the equation $y = 4x - 1$. Express the corresponding line in parametric form and vector form.

- When $x = t$, we must have $y = 4t - 1$. Thus,

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 4t - 1 \end{bmatrix}$$

So the parametric form will be

$$\begin{aligned} x &= t \\ y &= 4t - 1 \end{aligned}$$

and the vector form will be

$$\mathbf{x} = \begin{bmatrix} t \\ 4t - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

(b) Consider the equation $7x + 5y = 4$. Express the corresponding line in parametric form and vector form.

- Let $x = t$, then we must have $7t + 5y = 4$, or $y = \frac{4}{5} - \frac{7}{5}t$. Thus

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ \frac{4}{5} - \frac{7}{5}t \end{bmatrix}$$

The parametric form is thus

$$\begin{aligned} x &= t \\ y &= \frac{4}{5} - \frac{7}{5}t \end{aligned}$$

and the vector form is

$$\mathbf{x} = \begin{bmatrix} t \\ \frac{4}{5} - \frac{7}{5}t \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{5} \end{bmatrix} + t \begin{bmatrix} 1 \\ \frac{7}{5} \end{bmatrix}$$

13. Let $\mathbf{p} = (7,7,7)$, $\mathbf{q} = (4,0,2)$ and $\mathbf{r} = (0,1,-1)$. Give the vector equation of the plane passing through $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$.

We begin by constructing two directions living in the plane

$$\begin{aligned} \mathbf{d}_1 &= \mathbf{q} - \mathbf{p} = (4,0,2) - (7,7,7) = (-3,-7,-5) \\ \mathbf{d}_2 &= \mathbf{r} - \mathbf{p} = (0,1,-1) - (7,7,7) = (-7,-6,-8) \end{aligned}$$

The plane can now be constructed as the set of vectors that can be reached by starting at $\mathbf{p}$ and heading off in the directions $\mathbf{d}_1$ and $\mathbf{d}_2$:

$$\begin{aligned} P &= \{\mathbf{p} + s\mathbf{d}_1 + t\mathbf{d}_2 \mid s,t \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + s \begin{bmatrix} -3 \\ -7 \\ -5 \end{bmatrix} + t \begin{bmatrix} -7 \\ -6 \\ -8 \end{bmatrix} \mid s,t \in \mathbb{R} \right\} \end{aligned}$$

and so a typical point in the plane will have the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} + s \begin{bmatrix} -3 \\ -7 \\ -5 \end{bmatrix} + t \begin{bmatrix} -7 \\ -6 \\ -8 \end{bmatrix}$$