Math 3013
WebAssign Problems #4

1. Consider $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$. Determine whether the given matrix is in Row Echelon Form or Reduced Row Echelon Form.

- We begin by identifying the pivots in each row (underlined).

$$\begin{bmatrix} \underline{1} & 0 & 1 \\ 0 & 0 & \underline{4} \\ 0 & \underline{1} & 0 \end{bmatrix}$$

Since the pivot in the third row occurs to the left of the pivot in the second row, the matrix is **not** in Row Echelon Form. The matrix is not in Reduced Row Echelon Form either.

2. Consider $\begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Determine whether the given matrix is in Row Echelon Form or Reduced Row Echelon Form.

- Again we begin by underlining the pivots in each row.

$$\begin{bmatrix} 0 & \underline{1} & 2 & 0 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix}$$

Since the pivot of the upper row is to the left of the pivot in the bottom row, the matrix is in Row Echelon Form. It is also in Reduced Row Echelon Form since (i) it is in Row Echelon Form, (ii) all of its pivots are equal to 1, and (iii) directly above and below the pivots only 0's occur.

3. Consider $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Determine whether the given matrix is in Row Echelon Form or Reduced Row Echelon Form.

- Underlining the pivots we see

$$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & \underline{1} & 0 \\ \underline{1} & 0 & 0 \end{bmatrix}$$

Noting that the pivots of the upper rows lie off to the right of the pivots of lower rows, we conclude the matrix is **not** in Row Echelon Form. The matrix is not in Reduced Row Echelon Form either.

4. Use elementary row operations to reduce the given matrix to Row Echelon Form and Reduced Row Echelon Form.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- The first step is to get the left most pivot in the upper left hand corner. This can be done by interchanging the first and third rows:

$$\xrightarrow{R_1 \longleftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (R.E.F.)$$

Note that this last matrix is in Row Echelon Form. To be in Reduced Row Echelon Form, we need (i) it to be in Row Echelon Form, (ii) all of its pivots are equal to 1, and (iii) directly above and below the pivots only 0's occur. All but the last condition are satisfied. To satisfy (iii) we need to clear out the entries above the pivots in the lower rows. Thus,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

fixes the third column, and one more row operation will fix the second column:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the Reduced Row Echelon Form of the original matrix.

5. The **rank** of a matrix is the number of non-zero rows when the matrix is in Row Echelon Form. Determine the rank of each of the following matrices.

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

- A single row operation will cast this matrix into Row Echelon Form:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (R.E.F.)$$

  Since each row of the R.E.F. is non-zero, the rank of the matrix is 3.

(b) $\begin{bmatrix} 9 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- This matrix is already in Row Echelon Form. It has two non-zero rows. Therefore, its rank is 2.

(c) $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- This matrix is already in Row Echelon Form. It has two non-zero rows. Therefore, its rank is 2.

(d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- This matrix is in Row Echelon Form (even though it doesn't have any pivots). Since there are no non-zero rows, its rank is 0.

(e) $\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 0 & 1 \end{bmatrix}$

- This matrix is quickly converted to Row Echelon Form:

$$\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 1 & 8 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (R.E.F.)$$

  Since the R.E.F. has two non-zero rows, the rank of the matrix is 2.

(f) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

- This matrix is readily converted to Row Echelon Form:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (R.E.F.)$$

  Since the R.E.F. has 3 non-zero rows, the rank of the matrix is 3.

(g) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

- First we'll need to row reduce this matrix to Row Echelon Form.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 + 2R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad (R.E.F.)$$

  Since the matrix in R.E.F. has 3 non-zero rows, its rank is 3.

(h) $\begin{matrix} 2 & 1 & 3 & 8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{matrix}$

- This matrix is aready in Row Echelon Form. Since it has 3 non-zero rows, its rank is 3.

6. Solve the following system of equations using Row Reduction.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 19 \\ 2x_1 - x_2 + x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 8 \end{aligned}$$

- The augmented matrix for this system is

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 19 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 8 \end{array}\right]$$

We need to row reduce this augmented matrix to its Reduced Row Echelon Form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 19 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 8 \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 19 \\ 0 & -5 & 7 & -38 \\ 0 & -9 & 13 & -68 \end{array}\right]$$

$$\xrightarrow{R_2 \to -\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 19 \\ 0 & 1 & -\frac{7}{5} & \frac{38}{5} \\ 0 & -9 & 13 & -68 \end{array}\right] \xrightarrow{R_3 \to R_3 + 9R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 19 \\ 0 & 1 & -\frac{7}{5} & \frac{38}{5} \\ 0 & 0 & \frac{2}{5} & \frac{2}{5} \end{array}\right] \quad (R.E.F.)$$

$$\xrightarrow{R_3 \to \frac{5}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 19 \\ 0 & 1 & -\frac{7}{5} & \frac{38}{5} \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\substack{R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 + \frac{7}{5}R_3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 22 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad (R.R.E.F.)$$

Note that each column of the left hand side of the augmented matrix has a pivot. This means that there are no free variables and so we'll have a unique solution. Coverting back to equations we see our solution to be

$$\begin{aligned} x_1 &= 4 \\ x_2 &= 9 \\ x_3 &= 1 \end{aligned}$$

7. Solve the following system of equations using Row Reduction.

$$\begin{aligned} -x_1 + 3x_2 - 2x_3 + 4x_4 &= 0 \\ 2x_1 - 6x_2 + x_3 - 2x_4 &= -3 \\ x_1 - 3x_2 + 4x_3 - 8x_4 &= 2 \end{aligned}$$

- The augmented matrix for this system is

$$[\mathbf{A} \mid \mathbf{b}] = \left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right]$$

, row echelon form: $\begin{matrix} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$ We'll row reduce this matrix to its Reduced Row Echelon Form:

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \xrightarrow[R_3 \to R_3 + R_1]{\substack{R_1 \to -R_1 \\ R_2 \to R_2 + 2R_1}} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right]$$

$$\xrightarrow[R_3 \to \frac{1}{2}R_3]{R_2 \to -\frac{1}{3}R_2} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{cccc|c} 1 & -3 & 2 & -4 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc|c} 1 & -3 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad (R.R.E.F.)$$

Note that columns 2 and 4 of the augmented matrix in Reduced Row Echelon Form have no pivots. This means that x_2 and x_4 should be interpreted as free variables in solution. Writing down the equations corresponding to the augmented matrix in R.R.E.F.

$$\begin{aligned} x_1 - 3x_2 &= -2 \\ x_3 - 2x_4 &= 1 \\ 0 &= 0 \end{aligned}$$

and then move the free variables x_2 and x_4 to the right hand side. This allows us to express the fixed variables (x_1 and x_3) in terms of the free variables.

$$\begin{aligned} x_1 &= -2 + 3x_2 \\ x_3 &= 1 + 2x_4 \end{aligned}$$

Next we write down a typical solutions vector, substituting where we can for the fixed variables:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 + 3x_2 \\ x_2 \\ 1 + 2x_4 \\ x_4 \end{bmatrix}$$

Expanding the vector on the right hand side in terms of the free parameters:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

which the answer WebAssign expects (after relabeling the free parameters : $x_2 \to s$ and $x_4 \to t$).