Math 3013
WebAssign Problems #5

1. Theorem : If $\mathbf{A} = \begin{matrix} a & b \\ c & d \end{matrix}$, then $\mathbf{A}$ is invertible if $ad - bc \neq 0$, in which case

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then $\mathbf{A}$ is not invertible.

Find the inverse (if it exists) of the given matrix $\mathbf{A} = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$.

- Let's first prove the theorem. Our procedure for finding $\mathbf{A}^{-1}$ starts with row reducing the adjoined matrix $[\mathbf{A} \mid \mathbf{I}]$ to reduced row echelon form. So let's start with

$$[\mathbf{A} \mid \mathbf{I}] = \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

assuming $ad - cb \neq 0$ and apply our row reduction techniques.

$$\begin{aligned} &\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - \frac{c}{a}R_1} \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \\ &\xrightarrow[R_2 \to \frac{a}{ad-bc}R_2]{R_1 \to \frac{1}{a}R_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{a}{ad-bc}\left(-\frac{c}{a}\right) & \frac{a}{ad-bc} \end{array}\right] \\ = &\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{ad-bc}{ad} \end{array}\right] \xrightarrow{R_1 \to R_1 - \frac{b}{a}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{ad-bc}{ad} \end{array}\right] \end{aligned}$$

Note that in the R.R.E.F. matrix so obtained, the left hand side is the identity matrix. According to the procedure of Zoom Lecture 8, this means that the right hand side is $\mathbf{A}^{-1}$. Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{ad-bc}{ad} \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If, on the other hand, $ad - bc = 0$, then this row reduction can not be carried out to completion, since at some point we have to divide by $ad - bc = 0$. Thus, the right hand side of $[\mathbf{A} \mid \mathbf{I}]$ will not be row reducible to the identity matrix, and so $\mathbf{A}^{-1}$ will not exist.

- Now let's calculate the inverse of $\mathbf{A} = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix}$.

  One way to do this is to apply the theorem we just proved: Setting $a = 5$, $b = 19$, $c = 1$ and $d = 4$, we find

$$ad - bc = (4)(2) - (7)(1) = 1 \neq 0$$

So $\mathbf{A}$ is invertible and its inverse is

$$\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$$

A second way to find $\mathbf{A}^{-1}$ is just to apply our row reduction procedure:

$$
\begin{aligned}
[\mathbf{A} \mid \mathbf{I}] &= \left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 4 & 7 & 1 & 0 \end{array}\right] \\
& \xrightarrow{R_2 \to R_2 - 4R_1} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -4 \end{array}\right] \\
& \xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 4 \end{array}\right] \\
& \xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -7 \\ 0 & 1 & -1 & 5 \end{array}\right] \\
&= \left[\begin{array}{cc|cc} 1 & 0 & 2 & -7 \\ 0 & 1 & -1 & 4 \end{array}\right]
\end{aligned}
$$

Since the left hand side row reduces to the 2×2 identity matrix, the inverse of $\mathbf{A}$ is the right hand side of the R.R.E.F.:

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$$

2. Find the inverse of $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$.

- We'll proceed as in Problem 1 (without proving the theorem again).

  In this case, we have $a = 3$, $b = 4$, $c = 6$ and $d = 8$ and so

$$ad - bc = (3)(8) - (4)(6) = 24 - 24 = 0$$

and so the theorem says that $\mathbf{A}^{-1}$ does not exist.

On the other hand, if we row reduce $[\mathbf{A} \mid \mathbf{I}]$ to its Reduced Row Echelon Form

$$
\begin{aligned}
& \left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 6 & 8 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \\
& \xrightarrow{R_1 \to \frac{1}{3}R_1} \left[\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \qquad (RREF \text{ of } [\mathbf{A} \mid \mathbf{I}])
\end{aligned}
$$

Since the left hand side of the RREF of $[\mathbf{A} \mid \mathbf{I}]$ is not the identity matrix $\mathbf{A}$ is not invertible.

4. Find the inverse of $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

- We'll use row reduction to find $\mathbf{A}^{-1}$:

$$
\begin{aligned}
[\mathbf{A} \mid \mathbf{I}] &= \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \\
& \xrightarrow{R_1 \longleftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \qquad (RREF)
\end{aligned}
$$

The left hand side of the RREF is the identity matrix; therefore, the right hand side of the RREF is the inverse of $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1} = \left[\begin{array}{|ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]$$

(Note that $\mathbf{A}^{-1} = \mathbf{A}$ for this matrix.)

5. Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

- We'll use row reduction to find $\mathbf{A}^{-1}$:

$$
\begin{aligned}
[\mathbf{A} \mid \mathbf{I}] \quad &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_2 \to R_2 - cR_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad (RREF)
\end{aligned}
$$

The left hand side of the RREF is the identity matrix; therefore, the right hand side of the RREF is the inverse of $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

6. Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix}$

- We'll use row reduction to find $\mathbf{A}^{-1}$:

$$
\begin{aligned}
[\mathbf{A} \mid \mathbf{I}] \quad &= \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right] \\
&\xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right] \\
&\xrightarrow{R_1 \to R_1 - 5R_2} \left[\begin{array}{cc|cc} 1 & 0 & -4 & 5 \\ 0 & 1 & 1 & -1 \end{array}\right] \quad (RREF)
\end{aligned}
$$

The left hand side of the RREF is the identity matrix; therefore, the right hand side of the RREF is the inverse of $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1} = \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}$$

7. Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$

- We'll use row reduction to find $\mathbf{A}^{-1}$:

$$
\begin{aligned}
[\mathbf{A} \mid \mathbf{I}] \quad &= \left[\begin{array}{cc|cc} 1 & a & 1 & 0 \\ -a & 1 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_2 \to R_2 + aR_1} \left[\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & 1+a^2 & a & 1 \end{array}\right] \\
&\xrightarrow{R_2 \to \frac{1}{1+a^2} R_2} \left[\begin{array}{cc|cc} 1 & a & 1 & 0 \\ 0 & 1 & \frac{a}{1+a^2} & \frac{1}{1+a^2} \end{array}\right] \\
&\xrightarrow{R_1 \to R_1 - aR_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 - \frac{a^2}{1+a^2} & -\frac{a}{1+a^2} \\ 0 & 1 & \frac{a}{1+a^2} & \frac{1}{1+a^2} \end{array}\right] \quad (RREF)
\end{aligned}
$$

The left hand side of the RREF is the identity matrix; therefore, the right hand side of the RREF is the inverse of $\mathbf{A}^{-1}$:

$$\mathbf{A}^{-1} = \left| \begin{matrix} 1 - \frac{a^2}{1-a^2} & \frac{a}{1-a^2} \\ \frac{a}{1-a^2} & \frac{1}{1-a^2} \end{matrix} \right. = \frac{1}{1+a^2} \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}$$

8. Find the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$

- We'll use row reduction to find $\mathbf{A}^{-1}$:

$$\begin{aligned}
[\mathbf{A} \mid \mathbf{I}] &= \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \\
&\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 7 & 2 & 1 & -2 & 0 \\ 0 & -4 & 1 & 0 & -2 & 1 \end{array}\right] \\
&\xrightarrow{R_3 \to R_3 - \frac{4}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 7 & 2 & 1 & -2 & 0 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} & -\frac{6}{7} & 1 \end{array}\right] \\
&\xrightarrow[R_3 \to -7R_3]{R_3 \to \frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right] \\
&\xrightarrow[R_2 \to R_2 - \frac{2}{7}R_3]{R_1 \to R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 4 & 7 & -7 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right] \\
&\xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 3 & -3 \\ 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right] \quad (RREF)
\end{aligned}$$

The left hand side of the RREF is the identity matrix, therefore the right hand side of the RREF is the inverse of $\mathbf{A}$:

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix}$$