Math 3013
WebAssign Problems #6

1. Consider $\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}$,. Find bases for $RowSp(\mathbf{A})$, $ColSp(\mathbf{A})$, and $Null(\mathbf{A})$.

- We begin by row reducing $\mathbf{A}$ to R.R.E.F.

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow[R_2 \to \frac{1}{2}R_2]{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$
$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -\frac{7}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad (R.R.E.F.)$$

A basis for $RowSp(\mathbf{A})$ is given by the non-zero rows of any R.E.F. of $\mathbf{A}$. Thus,

$$\text{basis for } RowSp(\mathbf{A}) = \left\{ \left[1, 0, -\frac{7}{2}\right] , \left[0, 1, \frac{1}{2}\right] \right\}$$

A basis for $ColSp(\mathbf{A})$ is given by the columns of $\mathbf{A}$ that correspond to the columns of $R.E.F.(\mathbf{A})$ that have pivots. Our matrix in R.R.E.F. has pivots in its first two columns; so we can use columns 1 and 2 of $\mathbf{A}$:

$$\text{basis for } ColSp(\mathbf{A}) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$$

A basis for $Null(\mathbf{A})$ is given by the constant vectors that occur in the hyperplane form of the solution to $\mathbf{Ax} = \mathbf{0}$.

$$\begin{aligned}
R.R.E.F.(\mathbf{A}) &= \begin{bmatrix} 1 & 0 & -\frac{7}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad R.R.E.F.([\mathbf{A} \mid \mathbf{0}]) = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{7}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\
&\Rightarrow \left\{ \begin{array}{c} x_1 - \frac{7}{2}x_3 = 0 \\ x_2 + \frac{1}{2}x_3 = 0 \\ 0 = 0 \end{array} \right. \\
&\Rightarrow \left\{ \begin{array}{c} x_1 = \frac{7}{2}x_3 \\ x_2 = -\frac{1}{2}x_3 \end{array} \right. \\
&\Rightarrow \quad \mathbf{x} = \begin{bmatrix} \frac{7}{2}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}
\end{aligned}$$

Thus

$$\text{basis for } Null(\mathbf{A}) = \left\{ \begin{bmatrix} \frac{7}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$$

2. Consider $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$. Find bases for $RowSp(\mathbf{A})$, $ColSp(\mathbf{A})$, and $Null(\mathbf{A})$.

- A quick row reduction calculation reveals

$$R.R.E.F.(\mathbf{A}) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this R.R.E.F. we see (as in Problem 1)

$$\text{basis for } RowSp(\mathbf{A}) = \{[1,0,1,0]\ ,\ [0,1,-1,0]\ ,\ [0,0,0,1]\}$$

(since all three rows of the R.R.E.F. are non-zero).

$$\text{basis for } ColSp(\mathbf{A}) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(since columns $1, 2$, and 4 of the R.R.E.F. contain pivots).

And

$$\begin{aligned}
R.R.E.F.([\mathbf{A} \mid \mathbf{0}]) &= \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \\
&\Rightarrow \left\{ \begin{array}{c} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \\ x_4 = 0 \end{array} \right. \\
&\Rightarrow \left\{ \begin{array}{c} x_1 = -x_3 \\ x_2 = x_3 \\ x_4 = 0 \end{array} \right. \\
&\Rightarrow \quad \mathbf{x} = \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}$$

and so

$$\text{basis for } Null(\mathbf{A}) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

3. Consider $\mathbf{A} = \begin{bmatrix} 2 & -4 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 3 \\ 1 & -2 & 1 & 4 & 4 \end{bmatrix}$. Find bases for $RowSp(\mathbf{A})$, $ColSp(\mathbf{A})$, and $Null(\mathbf{A})$.

- A quick row reduction calculation reveals

$$R.R.E.F.(\mathbf{A}) = \begin{bmatrix} 1 & -2 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 3 & \frac{7}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$\text{basis for } RowSp(\mathbf{A}) = \left\{ \left[1, -2, 0, 1, \frac{1}{2}\right] \ , \ \left[0, 0, 1, 3, \frac{7}{2}\right] \right\}$$

(the two non-zero rows of $R.R.E.F.(\mathbf{A})$),

$$\text{basis for } ColSp(\mathbf{A}) = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(since columns 1 and 3 of $R.R.E.F.(\mathbf{A})$ have pivots, we can use columns 1 and 3 of $\mathbf{A}$ as a basis for $ColSp(\mathbf{A})$),

$$
\begin{aligned}
R.R.E.F\left(\left[\mathbf{A} \mid \mathbf{0}\right]\right) &= \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 3 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\
&\Rightarrow \left\{\begin{array}{r} x_1 - 2x_2 + x_4 + \frac{1}{2}x_5 = 0 \\ x_3 + 3x_4 + \frac{7}{2}x_5 = 0 \\ 0 = 0 \end{array}\right. \\
&\Rightarrow \left\{\begin{array}{c} x_1 = 2x_2 - x_4 - \frac{1}{2}x_5 \\ x_3 = -3x_4 - \frac{7}{2}x_5 \end{array}\right. \\
&\Rightarrow \quad \mathbf{x} = \left[\begin{array}{c} 2x_2 - x_4 - \frac{1}{2}x_5 \\ x_2 \\ -3x_4 - \frac{7}{2}x_5 \\ x_4 \\ x_5 \end{array}\right] = x_2 \left[\begin{array}{c} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right] + x_4 \left[\begin{array}{c} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{array}\right] + x_5 \left[\begin{array}{c} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{array}\right]
\end{aligned}
$$

and so

$$
\text{basis for } Null\left(\mathbf{A}\right) = \left\{ \left[\begin{array}{c} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} -1 \\ 0 \\ -3 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} -\frac{1}{2} \\ 0 \\ -\frac{7}{2} \\ 0 \\ 1 \end{array}\right] \right\}
$$

4. Find a basis for the span of $\left[\begin{array}{c} 1 \\ -1 \\ 0 \end{array}\right]$, $\left[\begin{array}{c} -1 \\ 0 \\ 1 \end{array}\right]$, and $\left[\begin{array}{c} 0 \\ 1 \\ -1 \end{array}\right]$.

- We write the given vectors as the rows of a matrix $\mathbf{A}$ and row reduce to R.E.F.

$$
\mathbf{A} = \left[\begin{array}{ccc} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] = R.R.E.F.\left(\mathbf{A}\right)
$$

The two non-zero rows of $R.R.E.F.\left(\mathbf{A}\right)$ will provide a basis for the span of the original three vectors:

$$
\text{basis} = \left\{ \left[\begin{array}{c} 1 \\ 0 \\ -1 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ -1 \end{array}\right] \right\}
$$

(Since the vectors were originally presented as column vectors, we present our basis vectors as column vectors as well.)

5. Find a basis for the span of $\left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$, $\left[\begin{array}{c} 1 \\ 2 \\ 0 \end{array}\right]$, $\left[\begin{array}{c} 0 \\ 1 \\ 1 \end{array}\right]$, and $\left[\begin{array}{c} 2 \\ 1 \\ 2 \end{array}\right]$

- We write the given vectors as the rows of a matrix $\mathbf{A}$ and row reduce to R.E.F.

$$
\mathbf{A} = \left[\begin{array}{ccc} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] = R.R.E.F.\left(\mathbf{A}\right)
$$

We can use the three non-zero rows of $R.R.E.F.\left(\mathbf{A}\right)$ as a basis for the span of the original vectors:

$$
\text{basis} = \left\{ \left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] \right\}
$$

(Since the vectors were originally presented as column vectors, we present our basis vectors as column vectors as well.)

6. Consider $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. Find the rank and nullity ($= \dim(Null\,(\mathbf{A}))$ of $\mathbf{A}$.

- We row reduce $\mathbf{A}$ to Row Echelon Form
$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = R.E.F.\,(\mathbf{A})$$
Since R.E.F.($\mathbf{A}$) has two pivots
$$Rank\,(\mathbf{A}) = 2$$
Since R.E.F.($\mathbf{A}$) has only one column without a pivot
$$Nullity\,(\mathbf{A}) = 1$$

7. $\mathbf{A} = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}$. Find the rank and nullity ($= \dim(Null\,(\mathbf{A}))$ of $\mathbf{A}$.

- We row reduce $\mathbf{A}$ to Row Echelon Form
$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \xrightarrow{\text{row reduction}} \begin{bmatrix} 1 & 0 & -\frac{7}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} = R.E.F.\,(\mathbf{A})$$
Since R.E.F.($\mathbf{A}$) has two pivots
$$Rank\,(\mathbf{A}) = 2$$
Since R.E.F.($\mathbf{A}$) has one column without a pivot
$$Nullity\,(\mathbf{A}) = 1$$

8. Consider $B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$. Show that $\mathbf{w}$ is in span(B) and find the coordinate vector $\mathbf{w}_B$ of $\mathbf{w}$ with respect to the basis B.

- We want to express $\mathbf{w}$ as a linear combination of the vectors in B:
$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$$
The coordinate vector $\mathbf{w}_B$ for $\mathbf{w}$ with respect to the basis B will then be $\mathbf{w}_B = [c_1, c_2]$. Looking at the above vector equation, component-by-component, we get
$$\begin{aligned} c_1 + c_2 &= 1 \\ 2c_1 &= 6 \\ -c_2 &= 2 \end{aligned}$$
We readily see from these equations that
$$\begin{aligned} c_1 &= 3 \\ c_2 &= -2 \end{aligned}$$
Thus,
$$\mathbf{w} = 3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Thus, the coordinate vector $\mathbf{w}_B$ for $\mathbf{w}$ with respect to the basis B will be

$$\mathbf{w}_B = [3, -2]$$