Math 3013
WebAssign Problem Set #7

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation corresponding to $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$. Find $T_{\mathbf{A}}(\mathbf{u})$ and $T_{\mathbf{A}}(\mathbf{v})$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

-

$$T_{\mathbf{A}}(\mathbf{u}) = \mathbf{A}\mathbf{u} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$
$$T_{\mathbf{A}}(\mathbf{v}) = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the matrix transformation corresponding to $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}$. Find $T_{\mathbf{A}}(\mathbf{u})$ and $T_{\mathbf{A}}(\mathbf{v})$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

-

$$T_{\mathbf{A}}(\mathbf{u}) = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix}$$
$$T_{\mathbf{A}}(\mathbf{v}) = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ -5 \end{bmatrix}$$

3. Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3 : T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x + 2y \\ 3x - 4y \end{bmatrix}$.

- We have

$$\mathbf{A}_T = \begin{bmatrix} \uparrow & \uparrow \\ T([1,0]) & T([0,1]) \\ \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 3 & -4 \end{bmatrix}$$

4. Find the standard matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2 : T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - y + z \\ 2x + y - 3z \end{bmatrix}$

- We have

$$\begin{aligned} \mathbf{A}_T &= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T([1,0,0]) & T([0,1,0]) & T([0,0,1]) \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix} \end{aligned}$$

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2 : T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$ and $S : \mathbb{R}^2 \to \mathbb{R}^2 : S\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 2y_1 \\ -y_2 \end{bmatrix}$.

(a) Calculate $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ and then $\mathbf{A}_{S \circ T}$ by direct substitution.

- $$\begin{aligned} S \circ T\left(\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]\right) &= S\left(\left[\begin{array}{c} x_1 - x_2 \\ x_1 + x_2 \end{array}\right]\right) \\ &= \left[\begin{array}{c} 2\left(x_1 - x_2\right) \\ -\left(x_1 + x_2\right) \end{array}\right] \\ &= \left[\begin{array}{c} 2x_1 - 2x_2 \\ -x_1 - x_2 \end{array}\right] \end{aligned}$$

  and so

  $$\begin{aligned} \mathbf{A}_{S\circ T} &= \left[\begin{array}{cc} \uparrow & \uparrow \\ \left(S \circ T\right)\left([1,0]\right) & \left(S \circ T\right)\left([0,1]\right) \\ \downarrow & \downarrow \end{array}\right] \\ &= \left[\begin{array}{cc} 2 & -2 \\ -1 & -1 \end{array}\right] \end{aligned}$$

(b) Calculate $\mathbf{A}_{S\circ T}$ as $\mathbf{A}_S \mathbf{A}_T$.

- We have

  $$\begin{aligned} \mathbf{A}_T &= \left[\begin{array}{cc} \uparrow & \uparrow \\ T\left([1,0]\right) & T\left([0,1]\right) \\ \downarrow & \downarrow \end{array}\right] = \left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right] \\ \mathbf{A}_S &= \left[\begin{array}{cc} \uparrow & \uparrow \\ S\left([1,0]\right) & S\left([0,1]\right) \\ \downarrow & \downarrow \end{array}\right] = \left[\begin{array}{cc} 2 & 0 \\ 0 & -1 \end{array}\right] \end{aligned}$$

  and so

  $$\mathbf{A}_{S\circ T} = \mathbf{A}_S \mathbf{A}_T = \left[\begin{array}{cc} 2 & 0 \\ 0 & -1 \end{array}\right]\left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right] = \left[\begin{array}{cc} 2 & -2 \\ -1 & -1 \end{array}\right]$$

6. Prove that the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is column space of its matrix $\mathbf{A}_T$.

- We have

  $$\begin{aligned} \mathit{Range}\left(T\right) &= \left\{\mathbf{y} \in \mathbb{R}^m \mid \mathbf{y} = T\left(\mathbf{x}\right) \text{ for some } \mathbf{x} \in \mathbb{R}^n\right\} \\ \mathit{ColSp}\left(\mathbf{A}_T\right) &= \mathit{span}\left(\mathit{Col}_1\left(\mathbf{A}_T\right), \ldots, \mathit{Col}_n\left(\mathbf{A}_T\right)\right) \end{aligned}$$

  To show that these two sets are equal, we show that (i) each element of $\mathit{Range}\left(T\right)$ is an element of $\mathit{ColSp}\left(\mathbf{A}_T\right)$ and that (ii) each element of $\mathit{ColSp}\left(\mathbf{A}_T\right)$ is an element of $\mathit{Range}\left(T\right)$.

  (i) Suppose $\mathbf{y} \in \mathit{Range}\left(T\right)$. Then there exists $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{y} = T\left(\mathbf{x}\right)$. Expand $\mathbf{x}$ with respect to the standard basis of $\mathbb{R}^n$

  $$\begin{aligned} \mathbf{y} &= T\left(\mathbf{x}\right) \\ &= T\left(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n\right) \\ &= T\left(x_1\mathbf{e}_1\right) + \cdots + T\left(x_n\mathbf{e}_n\right) \\ &= x_1 T\left(\mathbf{e}_1\right) + \cdots + x_n T\left(\mathbf{e}_n\right) \\ &= x_1 \mathit{Col}_1\left(\mathbf{A}_T\right) + \cdots + x_n \mathit{Col}_n\left(\mathbf{A}_T\right) \\ &\in \mathit{span}\left(\mathit{Col}_1\left(\mathbf{A}_T\right), \ldots, \mathit{Col}_n\left(\mathbf{A}_t\right)\right) \\ &= \mathit{ColSp}\left(\mathbf{A}_T\right) \end{aligned}$$

  (ii) Suppose $\mathbf{y} \in \mathit{ColSp}\left(\mathbf{A}_T\right)$. Then

  $$\mathbf{y} = c_1 \mathit{Col}_1\left(\mathbf{A}_T\right) + \cdots + c_n \mathit{Col}_n\left(\mathbf{A}_T\right)$$

for some choice of coefficients $c_1, \ldots, c_n$. But then

$$\begin{aligned} \mathbf{y} &= c_1 T(\mathbf{e}_1) + \cdots + c_n T(\mathbf{e}_n) \\ &= T(c_1 \mathbf{e}_1) + \cdots + T(c_n \mathbf{e}_n) \\ &= T(c_1 \mathbf{e}_1 + \cdots + c_n \mathbf{e}_n) \\ &\in Range(T) \end{aligned}$$

Since $Range(T) \subseteq ColSp(\mathbf{A}_T)$ and $ColSp(\mathbf{A}_T) \subseteq Range(T)$

$$Range(T) = ColSp(\mathbf{A}_T)$$