Math 3013
WebAssign Problem Set #8

1. Compute the determinant of $\mathbf{A} = \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix}$

- Applying the cofactor expansion formula to the third column:

$$\begin{aligned} \det(\mathbf{A}) &= (3)(-1)^{1+3}\det\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} + (4)(-1)^{2+3}\det\begin{pmatrix} -4 & 1 \\ 1 & -1 \end{pmatrix} + (0)(-1)^{3+3}\det\begin{pmatrix} -4 & 1 \\ 2 & -2 \end{pmatrix} \\ &= (3)(-2+2) - (4)(4-1) + 0 \\ &= -12 \end{aligned}$$

2. Compute the determinant of $\mathbf{A} = \begin{bmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{bmatrix}$

- Applying the cofactor expansion formula to the third row:

$$\begin{aligned} \det(\mathbf{A}) &= 0 + (e)(-1)^{3+2}\det\begin{pmatrix} 0 & 0 \\ b & d \end{pmatrix} + 0 \\ &= 0 + 0 + 0 \end{aligned}$$

3. Compute the determinant of $\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 3 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix}$

- Applying the cofactor expansion formula to the third row:

$$\begin{aligned} \det(\mathbf{A}) &= 0 + (1)(-1)^{3+2}\det\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 6 \\ 1 & 2 & 1 \end{pmatrix} + 0 + 0 \\ &= -\det\begin{pmatrix} 1 & 0 & 3 \\ 2 & 3 & 6 \\ 1 & 2 & 1 \end{pmatrix} \end{aligned}$$

Applying the cofactor expansion formula to the first row (of the 3×3 matrix on the right)

$$\begin{aligned} \det(\mathbf{A}) &= -(1)(-1)^{1+1}\det\begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix} + 0 - (3)(-1)^{1+3}\det\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \\ &= -(1)(3-12) + 0 - (3)(4-3) \\ &= 9 - 3 \\ &= 6 \end{aligned}$$

- We could also calculate $\det(\mathbf{A})$ using its row reduction to R.E.F.:

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 3 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix} \xrightarrow[R_4 \to R_4 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 7 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 7 & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 2 & -2 \end{bmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 7 & 3 & 0 \\ 0 & 5 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 7 & 3 & 0 \\ 0 & 5 & 2 & -2 \end{bmatrix} \xrightarrow[R_4 - R_1]{R_3 - 7R_2} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{3}R_3} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -2 \end{bmatrix} \xrightarrow{R_4 \to R_4 - 2R_3} \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The last matrix is the R.E.F. of $\mathbf{A}$. We have

$$\det(\mathbf{A}) = (-1)^{\#\text{row interchanges}} \frac{1}{\text{product of row rescalings}} \det(R.E.F.(\mathbf{A}))$$

In the row reduction we used 1 row interchange, a row rescaling by a factor of $\frac{1}{3}$ and since the R.E.F. is upper triangular its determinant is just the product of its diagonal elements: thus,

$$\det(\mathbf{A}) = (-1)^1 \frac{1}{\frac{1}{3}} (1)(1)(1)(-2) = -6$$

4. Find all values of the parameter k such that $\mathbf{A} = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$ is invertible.

- The matrix $\mathbf{A}$ is invertible only if $\det(\mathbf{A}) \neq 0$. Applying the cofactor expansion formula to the first column:

$$\begin{aligned} \det(\mathbf{A}) &= (k)(-1)^{1+1} \det \begin{pmatrix} k+1 & 1 \\ -8 & k-1 \end{pmatrix} + 0 + (k)(-1)^{3+1} \det \begin{pmatrix} -k & 3 \\ k+1 & 1 \end{pmatrix} \\ &= (k)\left(k^2 - 1 + 8\right) + 0 + (k)(-k - 3k - 3) \\ &= k^3 - 4k^2 + 4k \\ &= k(k-2)^2 \end{aligned}$$

Since $\det(\mathbf{A}) = 0$ if and only if $k = 0$ or $k = 2$, the matrix $\mathbf{A}$ is invertible for all values of k except $k = 0$ and $k = 2$.

5. Use Cramer's Rule to solve

$$\begin{aligned} x + y &= 1 \\ x - y &= 2 \end{aligned}$$

- For this linear system, we have

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the following auxiliary matrices (each obtained from $\mathbf{A}$ by replacing a column of $\mathbf{A}$ with the column vector $\mathbf{b}$)

$$\mathbf{B}_1 = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \quad , \quad \mathbf{B}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Cramer's Rule tells us that the components of a solution vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are given by

$$x_1 = \frac{\det(\mathbf{B}_1)}{\det(\mathbf{A})} \quad , \quad x_2 = \frac{\det(\mathbf{B}_2)}{\det(\mathbf{A})}$$

Since

$$\begin{aligned} \det(\mathbf{A}) &= -1 - 1 = -2 \\ \det(\mathbf{B}_1) &= -1 - 2 = -3 \\ \det(\mathbf{B}_2) &= 2 - 1 = 1 \end{aligned}$$

we have

$$\begin{aligned} x_1 &= \frac{-3}{-2} = \frac{3}{2} \\ x_2 &= \frac{1}{-2} = -\frac{1}{2} \end{aligned}$$

6. User Cramer's Rule to solve

$$\begin{aligned} x + y - z &= 1 \\ x + y + z &= 2 \\ x - y &= 3 \end{aligned}$$

- We have

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \\ \mathbf{b} &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{aligned}$$

, determinant: 4and

$$\begin{aligned} \mathbf{B}_1 &= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{bmatrix} \\ \mathbf{B}_2 &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix} \\ \mathbf{B}_3 &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \end{aligned}$$

, determinant: 2, determinant: -3, determinant: 9One finds

$$\begin{aligned}\det(\mathbf{A}) &= 4\\ \det(\mathbf{B}_1) &= 9\\ \det(\mathbf{B}_2) &= -3\\ \det(\mathbf{B}_3) &= 2\end{aligned}$$

and so the components of a solution vector $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ are given by

$$\begin{aligned}x &= \frac{9}{4}\\ y &= \frac{-3}{4}\\ z &= \frac{1}{2}\end{aligned}$$

7. Use the formula $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A})$ to compute the inverse of $\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

- We first compute the entries of the cofactor matrix of $\mathbf{A}$

$$\begin{aligned}C_{11} &= (-1)^{1+1}\det(\mathbf{M}_{11}) = \det([3]) = 3\\ C_{12} &= (-1)^{1+2}\det(\mathbf{M}_{11}) = -\det([1]) = -1\\ C_{21} &= (-1)^{2+1}\det(\mathbf{M}_{21}) = -\det([-1]) = 1\\ C_{22} &= (-1)^{2+2}\det(\mathbf{M}_{22}) = \det([2]) = 2\end{aligned}$$

Thus,

$$\mathbf{C} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$$

Now the adjugate matrix $adj(\mathbf{A})$ of $\mathbf{A}$ is the transpose of the cofactor matrix

$$adj(\mathbf{A}) = \mathbf{C}^t = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

and

$$\det(\mathbf{A}) = 6 + 1 = 7$$

Thus,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A}) = \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

8. Use the formula $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj(\mathbf{A})$ to compute the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

- We first compute the entries of the cofactor matrix of $\mathbf{A}$:

$$\begin{aligned}C_{11} &= (-1)^{1+1}\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = 1\\ C_{12} &= (-1)^{1+2}\det\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0\\ C_{13} &= (-1)^{1+3}\det\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0\end{aligned}$$

$$\begin{aligned}
C_{21} &= (-1)^{2+1} \det \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = -1 \\
C_{22} &= (-1)^{2+2} \det \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} = 2 \\
C_{23} &= (-1)^{2+3} \det \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = 0
\end{aligned}$$

$$\begin{aligned}
C_{31} &= (-1)^{3+1} \det \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} = -2 \\
C_{32} &= (-1)^{3+2} \det \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = 0 \\
C_{33} &= (-1)^{3+3} \det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = 2
\end{aligned}$$

Using these cofactors we can readily calculate $\det(\mathbf{A})$ along any row or column. For example, applying the cofactor expansion formula to the third row of $\mathbf{A}$ yields

$$\det(\mathbf{A}) = 0 + 0 + (1)\, C_{33} = 2$$

And

$$adj\,(\mathbf{A}) = \mathbf{C}^t = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

and so

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} adj\,(\mathbf{A}) = \frac{1}{2} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$