Math 3013
WebAssign Problem Set #9

1. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$ compute: (a) the characteristic polynomial of $\mathbf{A}$, (b) the eigenvalues of $\mathbf{A}$, (c) a basis for each eigenspace of $\mathbf{A}$, and (d) the algebraic and geometric multiplicities of each eigenvalue.

(a) The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} p_{\mathbf{A}}(\lambda) &= \det(\mathbf{A} - \lambda \mathbf{I}) = \det\left(\begin{bmatrix} 1-\lambda & 3 \\ -2 & 6-\lambda \end{bmatrix}\right) = (1-\lambda)(6-\lambda) + 6 \\ &= \lambda^2 - 7\lambda + 12 \\ &= (\lambda - 3)(\lambda - 4) \end{aligned}$$

(b) The eigenvalues of $\mathbf{A}$ are the solutions of $p_{\mathbf{A}}(\lambda) = 0$; thus,

$$\text{eigenvalues of } \mathbf{A} \text{ are } \{3, 4\}$$

(c)

3-eigenspace

$$\begin{aligned} E_3 &= NullSp(\mathbf{A} - 3\mathbf{I}) = NullSp\begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix} = NullSp\begin{pmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} \\ &= span\left(\begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}\right) = span\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) \end{aligned}$$

$$\text{basis for 3-eigenspace} = \left\{\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$$

4-eigenspace

$$\begin{aligned} E_4 &= NullSp(\mathbf{A} - 4\mathbf{I}) = NullSp\begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix} = NullSp\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\ &= span\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \end{aligned}$$

$$\text{basis for 4-eigenspace} = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

(d) The algebraic multiplicity of an eigenvalue r of $\mathbf{A}$ is the number of factors of $(\lambda - r)$ in the characteristic polynomial $p_{\mathbf{A}}(\lambda)$. Thus,

$$\begin{aligned} \text{algebraic multiplicity of eigenvalue 3} &= 1 \\ \text{algebraic multiplicity of eigenvalue 4} &= 1 \end{aligned}$$

The geometric multiplicity of an eigenvalue r of $\mathbf{A}$ is the dimension of the corresponding eigenspace E_r - which is, by definition, the number of basis vectors for E_r. Thus,

$$\begin{aligned} \text{geometric multiplicity of eigenvalue 3} &= 1 \\ \text{geometric multiplicity of eigenvalue 4} &= 1 \end{aligned}$$

2. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ compute: (a) the characteristic polynomial of $\mathbf{A}$, (b) the eigenvalues of $\mathbf{A}$, (c) a basis for each eigenspace of $\mathbf{A}$, and (d) the algebraic and geometric multiplicities of each eigenvalue.

(a) We have

$$p_{\mathbf{A}}(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(-2-\lambda)(3-\lambda)$$

(b)

$$0 = p_{\mathbf{A}}(\lambda) \quad \Rightarrow \quad \lambda = 1, -2, 3 \qquad \text{(the eigenvalues of } \mathbf{A}\text{)}$$

- 1-eigenspace

$$\begin{aligned}
E_1 &= NullSp\left(\begin{bmatrix} 1-1 & 1 & 0 \\ 0 & -2-1 & 1 \\ 0 & 0 & 3-1 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 2 \end{bmatrix}\right) \\
&= NullSp\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \\
&\Rightarrow \quad \text{basis vector } = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

(c)

-2-eigenspace

$$\begin{aligned}
E_{-2} &= NullSp\left(\begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= span\left(\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}\right) \\
&\Rightarrow \quad \text{basis vector } = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}
\end{aligned}$$

3-eigenspace

$$\begin{aligned}
E_3 &= NullSp\left(\begin{bmatrix} -2 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 0 & -\frac{1}{10} \\ 0 & 1 & -\frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= span\left(\begin{bmatrix} \frac{1}{10} \\ \frac{1}{5} \\ 1 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}\right) \\
&\Rightarrow \quad \text{basis vector } = \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}
\end{aligned}$$

(d) The algebraic multiplicity of an eigenvalue r of $\mathbf{A}$ is the number of factors of $(\lambda - r)$ in the characteristic polynomial $p_{\mathbf{A}}(\lambda)$.

The geometric multiplicity of an eigenvalue r of $\mathbf{A}$ is the dimension of the corresponding eigenspace E_r - which is, by definition, the number of basis vectors for E_r. Thus,

eigenvalue	algebraic multiplicity	geometric multiplicity
1	1	1
-2	1	1
3	1	1

3. For the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ compute: (a) the characteristic polynomial of $\mathbf{A}$, (b) the eigenvalues of $\mathbf{A}$, (c) a basis for each eigenspace of $\mathbf{A}$, and (d) the algebraic and geometric multiplicities of each eigenvalue.

(a)

$$
\begin{aligned}
p_{\mathbf{A}}(\lambda) &= \det\left(\begin{bmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}\right) = -\lambda^3 - \lambda^2 \\
&= -\lambda^2 (1-\lambda) \\
&= -(0-\lambda)^2 (1-\lambda)
\end{aligned}
$$

(b) We have two eigenvalues

$$
\begin{aligned}
\lambda &= 0 \text{ with algebraic multiplicity } 2 \\
\lambda &= 1 \text{ with algebraic multiplicity } 1
\end{aligned}
$$

(c)

0-eigenspace

$$
\begin{aligned}
E_0 &= NullSp\left(\begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= span\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right) \\
&\Rightarrow \quad \text{basis} = \left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}
\end{aligned}
$$

1-eigenspace

$$
\begin{aligned}
E_1 &= NullSp\left(\begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) \\
&\Rightarrow \quad \text{basis} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}
\end{aligned}
$$

(d) The algebraic multiplicity of an eigenvalue r of $\mathbf{A}$ is the number of factors of $(\lambda - r)$ in the characteristic polynomial $p_{\mathbf{A}}(\lambda)$.

The geometric multiplicity of an eigenvalue r of $\mathbf{A}$ is the dimension of the corresponding eigenspace E_r - which is, by definition, the number of basis vectors for E_r. Thus,

eigenvalue	algebraic multiplicity	geometric multiplicity
0	2	1
1	1	1

4. For the matrix $\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$ compute: (a) the characteristic polynomial of $\mathbf{A}$, (b) the eigenvalues of $\mathbf{A}$, (c) a basis for each eigenspace of $\mathbf{A}$, and (d) the algebraic and geometric multiplicities of each eigenvalue.

(a)

$$
p_{\mathbf{A}}(\lambda) = \det\begin{pmatrix} 4-\lambda & 0 & 1 \\ 2 & 3-\lambda & 2 \\ -1 & 0 & 2-\lambda \end{pmatrix} = -\lambda^3 + 9\lambda^2 - 27\lambda + 27
$$

(b) To identify the eigenvalues of $\mathbf{A}$ we need to completely factorize $p_{\mathbf{A}}(\lambda)$. Note that $\lambda = 3$ is a root of $p_{\mathbf{A}}(\lambda) = 0$, since

$$p_{\mathbf{A}}(3) = -27 + 81 - 81 + 27 = 0$$

This implies that $(\lambda - 3)$ divides $p_{\mathbf{A}}(\lambda)$. A polynomial division calculation shows that

$$\frac{-\lambda^3 + 9\lambda^2 - 27\lambda + 27}{\lambda - 3} = -(\lambda - 3)^2$$

Thus,

$$p_{\mathbf{A}}(\lambda) = -(\lambda - 3)(\lambda - 3)^2 = (\lambda - 3)^3$$

We thus have one eigenvalue $\lambda = 3$ with algebraic multiplicity 3.

(c)

3-eigenspace

$$\begin{aligned} E_3 &= NullSp\left(\begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{bmatrix}\right) = NullSp\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= span\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) \\ \text{basis for } E_3 &= \left\{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \end{aligned}$$

(d) The algebraic multiplicity of an eigenvalue r of $\mathbf{A}$ is the number of factors of $(\lambda - r)$ in the characteristic polynomial $p_{\mathbf{A}}(\lambda)$.

The geometric multiplicity of an eigenvalue r of $\mathbf{A}$ is the dimension of the corresponding eigenspace E_r - which is, by definition, the number of basis vectors for E_r. Thus,

eigenvalue	algebraic multiplicity	geometric multiplicity
3	3	2

5. Below is the diagonalization of a matrix $\mathbf{A}$ given in the form $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$. List the eigenvalues of $\mathbf{A}$ and bases for the corresponding eigenspace.

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

- The eigenvalues of $\mathbf{A}$ are the diagonal entries of the matrix $\mathbf{D}$. Thus,

$$\text{eigenvalues of } \mathbf{A} = \{4, 3\}$$

The basis vectors $\mathbf{v}_{\lambda=4}$, $\mathbf{v}_{\lambda=3}$ for the corresponding eigenspaces of $\mathbf{A}$ are the corresponding column vectors of $\mathbf{P}$, thus

$$\mathbf{v}_{\lambda=4} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad \mathbf{v}_{\lambda=3} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

6. Determine whether $\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ is diagonalizable, and if so, an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

- We have

$$\begin{aligned} p_{\mathbf{A}}(\lambda) &= \det\left(\begin{bmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix}\right) = (5-\lambda)^2 - 4 = \lambda^2 - 10\lambda + 21 \\ &= (\lambda - 3)(\lambda - 7) \end{aligned}$$

So the eigenvalues of $\mathbf{A}$ are $\lambda = 3, 7$.

Since $\mathbf{A}$ is a 2×2 matrix with 2 distinct eigenvalues, $\mathbf{A}$ must be diagonalizable.

3-eigenspace

$$E_3 = NullSp\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = span\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \quad \Rightarrow \quad \mathbf{v}_{\lambda=3} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

7-eigenspace

$$\begin{aligned} E_4 &= NullSp\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = NullSp\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\ &\Rightarrow \quad \mathbf{v}_{\lambda=7} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Thus,

$$\mathbf{P} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}$$

7. Determine whether $\mathbf{A} = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$ is diagonalizable, and if so, an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

- We have

$$\begin{aligned} p_{\mathbf{A}}(\lambda) &= \det\left(\begin{bmatrix} -3-\lambda & 4 \\ -1 & 1-\lambda \end{bmatrix}\right) = (-3-\lambda)(1-\lambda) + 4 \\ &= \lambda^2 + 2\lambda + 1 \\ &= (\lambda+1)^2 \end{aligned}$$

We thus have only one eigenvalue $\lambda = -1$ (occurring with algebraic multiplicity 2)

For $\mathbf{A}$ to be diagonalizable, we need two linearly independent eigenvectors.

$$E_{-1} = NullSp(\mathbf{A} - (-1)\mathbf{I}) = NullSp\left(\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}\right)$$

Since we have only one column without a pivot, we see that E_{-1} must be 1-dimensional. Thus, $\mathbf{A}$ has only one linearly independent eigenvector, and so $\mathbf{A}$ is **not diagonalizable**.

8. Determine whether $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is diagonalizable, and if so, an invertible matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

- We have

$$p_{\mathbf{A}}(\lambda) = \det\left(\begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & 0-\lambda \end{bmatrix}\right) = -\lambda^3 + 2\lambda^2 + \lambda - 2$$

To completely factorize $p_{\mathbf{A}}(\lambda)$, we first note that $\lambda = 2$ is a solution: for

$$p_{\mathbf{A}}(2) = -8 + 8 + 2 - 2 = 0$$

This means $(\lambda - 2)$ is a factor of $p_{\mathbf{A}}(\lambda)$, and a polynomial division calculation shows

$$\frac{-\lambda^3 + 2\lambda^2 + \lambda - 2}{\lambda - 2} = 1 - \lambda^2$$

Thus,

$$p_{\mathbf{A}}(\lambda) = (\lambda - 2)\left(1 - \lambda^2\right) = (\lambda - 2)(1 - \lambda)(1 + \lambda)$$

Hence, we have three distinct eigenvalues, $\lambda = 2, 1, -1$. As a 3×3 matrix with 3 distinct eigenvalues, $\mathbf{A}$ must be diagonalizable. We'll now find a basis vector for each eigenspace:

$$
\begin{aligned}
E_2 &= NullSp\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \\
\Rightarrow &\quad \mathbf{v}_{\lambda=2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
E_1 &= NullSp\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) \\
\Rightarrow &\quad \mathbf{v}_{\lambda=1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
E_{-1} &= NullSp\left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= span\left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right) = span\left(\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right) \\
\Rightarrow &\quad \mathbf{v}_{\lambda=-1} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}
\end{aligned}
$$

We thus can take

$$
\begin{aligned}
\mathbf{D} &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
\mathbf{P} &= \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{v}_{\lambda=2} & \mathbf{v}_{\lambda=1} & \mathbf{v}_{\lambda=-1} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & -2 \end{bmatrix}
\end{aligned}
$$