Math 3013
WebAssign Problem Set #10

1. Determine if the following vectors are an orthogonal set : $\mathbf{a} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$

- We have

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= -3 + 2 - 1 = -2 \\ \mathbf{a} \cdot \mathbf{c} &= 6 - 2 - 4 = 0 \\ \mathbf{b} \cdot \mathbf{c} &= -2 - 4 + 4 = -2 \end{aligned}$$

Since $0 \neq \mathbf{a} \cdot \mathbf{b}$ and $0 \neq \mathbf{b} \cdot \mathbf{c}$, these three vectors are not an orthogonal set of vectors.

2. Find the orthogonal complement $W^{\perp}$ of $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x + y - z = 0 \right\}$

- First, we need a basis for W. Solving $x + y - z = 0$, leads us to

$$\begin{aligned} W &= \left\{ \begin{bmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} \mid x_2, x_3 \in \mathbb{R} \right\} \\ &= span\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

So our basis for W is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Next, we have

$$\begin{aligned} W^{\perp} &= NullSp\left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \right) \xrightarrow{\text{row reduction}} NullSp\left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right) \\ &= span\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right) \end{aligned}$$

3. Let $W = span\left(\begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \end{bmatrix} , \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right)$. Find a basis for $W^{\perp}$.

- We need to find a basis for

$$\begin{aligned} W^{\perp} &= NullSp\left(\begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{bmatrix} \right) \xrightarrow{\text{row reduction}} NullSp\left(\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \right) \\ &= \left\{ \begin{bmatrix} -2x_3 + x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} \mid x_3, x_4 \in \mathbb{R} \right\} = span\left(\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

and so our basis for $W^{\perp}$ is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} , \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. Find the orthogonal projection of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on the subspace spanned by $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

- If $W = span(\mathbf{u}_1, \mathbf{u}_2)$, then

$$\begin{aligned} W^{\perp} &= NullSp\left(\begin{bmatrix} \leftarrow & \mathbf{u}_1 & \rightarrow \\ \leftarrow & \mathbf{u}_2 & \rightarrow \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 2 & -2 & 1 \\ -1 & 1 & 4 \end{bmatrix}\right) \\ &= NullSp\left(\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= span\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) \end{aligned}$$

So $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $W^{\perp}$. Noting that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, and so provide a basis for W, we can combine $\mathbf{w}$ with $\mathbf{u}_1$ and $\mathbf{u}_2$ to get a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{w}\}$ for $\mathbb{R}^3$. Next we find the coordinate vector $\mathbf{v}_B$ of $\mathbf{v}$ with respect to the basis B for $\mathbb{R}^3$. Using

$$\left[\begin{array}{ccc|c} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{w} & \mathbf{v} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}\right] \xrightarrow{\text{row reduction}} [\mathbf{I} \mid \mathbf{v}_B]$$

we find

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ -2 & 1 & 1 & 2 \\ 1 & 4 & 0 & 3 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{9} \\ 0 & 1 & 0 & \frac{13}{18} \\ 0 & 0 & 1 & \frac{3}{2} \end{array}\right]$$

Hence

$$\mathbf{v}_B = \begin{bmatrix} \frac{1}{9} \\ \frac{13}{18} \\ \frac{3}{2} \end{bmatrix}$$

Thus,

$$\mathbf{v} = \left(\frac{1}{9}\right)\mathbf{u}_1 + \left(\frac{13}{18}\right)\mathbf{u}_2 + \frac{3}{2}\mathbf{w}$$

The orthogonal projection of $\mathbf{v}$ onto W is then (since only $\mathbf{u}_1$ and $\mathbf{u}_2$ are basis vectors for W) is thus

$$\begin{aligned} \mathbf{v}_W &= \left(\frac{1}{9}\right)\mathbf{u}_1 + \left(\frac{13}{18}\right)\mathbf{u}_2 \\ &= \left(\frac{1}{9}\right)\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \left(\frac{13}{18}\right)\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix} \end{aligned}$$

5. Use the Gram-Schmidt Process to construct an othogonal basis from the basis

$\left\{ \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} , \mathbf{x}_2 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} , \mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \right\}$ of $\mathbb{R}^3$. Then normalize this orthogonal basis to obtain an orthonormal basis for $\mathbb{R}^3$.

- We set

$$\mathbf{o}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Next,

$$\begin{aligned} \mathbf{o}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{o}_1}{\mathbf{o}_1 \cdot \mathbf{o}_1}\mathbf{o}_1 \\ &= \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} - \left(\frac{0}{3}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \end{aligned}$$

And then

$$\begin{aligned} \mathbf{o}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{o}_1}{\mathbf{o}_1 \cdot \mathbf{o}_1}\mathbf{o}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{o}_2}{\mathbf{o}_2 \cdot \mathbf{o}_2}\mathbf{o}_2 \\ &= \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} - \left(\frac{5}{3}\right) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{18}{18}\right) \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \end{aligned}$$

So

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} , \begin{matrix} \frac{7}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{matrix} \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \right\}$$

is an orthogonal basis for $\mathbb{R}^3$.

Finally, to get an orthonormal basis for $\mathbb{R}^3$, we normalize the vectors of the othogonal basis so that they have unit length

$$\begin{aligned} \mathbf{n}_1 &= \frac{1}{\|\mathbf{o}_1\|}\mathbf{o}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix} \\ \mathbf{n}_2 &= \frac{1}{\|\mathbf{o}_2\|}\mathbf{o}_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\ \mathbf{n}_3 &= \frac{1}{\|\mathbf{o}_3\|}\mathbf{o}_3 = \sqrt{\frac{9}{16+4+4}} \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{6} \end{bmatrix} \end{aligned}$$

6. Given that $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ form a basis for a subspace W. Apply the Gram-Schmidt Process to find an orthogonal basis for W.

- We begin by setting

$$\mathbf{o}_1 = \mathbf{x}_1$$

and then

$$\begin{aligned} \mathbf{o}_2 &= \mathbf{x}_2 - \frac{\mathbf{o}_1 \cdot \mathbf{x}_2}{\mathbf{o}_1 \cdot \mathbf{o}_1}\mathbf{o}_1 \\ &= \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \left(\frac{3+4+0}{9+16+4} \right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{80}{29} \\ \frac{109}{29} \\ 2 \end{bmatrix} \end{aligned}$$

And so

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{80}{29} \\ \frac{109}{29} \\ 2 \end{bmatrix} \right\}$$

will be an orthogonal basis for W.