LECTURE 6

# Functions

Recall that a *function* (or *map*, or *mapping*) $f$ from a set $A$ to a set $B$ is rule that assigns to each element $a \in A$ exactly one element $b \in B$; $b$ is called the *image* of $a$ by $f$, or the *value* of $f$ at $a$, and is usually denoted $f(a)$. The set $A$ is called the *domain* of $f$ and the set $B$ is called the *range* of $f$.

We will frequently use the notation

$$f : A \to B$$

to indicate a function whose domain is the set $A$ and whose range is the set $B$.

## 1. Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two functions such that the range of $f$ coincides with the domain of $g$. Then the **composite** of $f$ and $g$ is the function $g \circ f : A \to C$ whose rule is

$$g \circ f(x) = g(f(x)) \quad , \quad \forall\, x \in A \quad .$$

EXAMPLE 6.1. Let $\mathbb{E}$ denote the set of even integers; $\mathbb{Z}$, the set of integers; and $\mathbb{N}$ the set of natural numbers. Let $f : \mathbb{E} \to \mathbb{Z}$ be the map defined by

$$f(e) = \frac{e}{2} \quad , \quad \forall\, e \in \mathbb{E}.$$

Let $g : \mathbb{Z} \to \mathbb{N}$ be the map defined by

$$g(z) = z^2 \quad , \quad \forall\, z \in \mathbb{Z} \quad .$$

Then the composite mapping $g \circ f : \mathbb{E} \to \mathbb{N}$ has the rule

$$e \mapsto (g \circ f)(e) = \left(\frac{e}{2}\right)^2 = \frac{e^2}{4} \quad .$$

Note that the composite function with the opposite order $f \circ g$ is *not defined*, since the domain $\mathbb{E}$ of $f$ is not contained in the range $\mathbb{N}$ of $g$.

EXAMPLE 6.2. Let

$$\begin{aligned} f &: \mathbb{Z} \to \mathbb{Z} \quad ; \quad f(n) = n - 1 \\ g &: \mathbb{Z} \to \mathbb{Z} \quad ; \quad g(n) = n^2 \end{aligned}$$

Then

$$\begin{aligned} (f \circ g)(n) &= f(g(n)) \\ &= f\left(n^2\right) \\ &= n^2 + 1 \quad , \end{aligned}$$

while

$$\begin{aligned} (g \circ f)(n) &= g(f(n)) \\ &= g(n-1) \\ &= (n-1)^2 \\ &= n^2 - 2n + 1 \quad . \end{aligned}$$

So even though both $f \circ g$ and $g \circ f$ are both well-defined, $f \circ g$ **is not the same function as** $g \circ f$.

We conclude that **the composite of two functions depends on the order in which they are composed**.

DEFINITION 6.3. *A function $f : A \to B$ is said to be* ***injective*** *(or* ***one-to-one****) if whenever $f(a) = f(a')$, $a = a'$.*

DEFINITION 6.4. *A function $f : A \to B$ is said to be* ***surjective*** *(or* ***onto****) if every element $b \in B$ is in the image of the $f$; i.e., $b = f(a)$ for some $a \in A$.*

DEFINITION 6.5. *A function $f : A \to B$ is said to be* ***bijective*** *(or a* ***bijection****) if $f$ is both injective and surjective.*

THEOREM 6.6. *A function $f : A \to B$ is bijective if and only if there exists a function $g : B \to A$ such that*

$$g \circ f = I_A \quad \text{and} \quad f \circ g = I_B \quad ,$$

*where $I_A : A \to A$ and $I_B : B \to B$ are, respectively, the identity maps of $A$ and $B$.*