Math 4023
Homework Set 3

1. Suppose that $\mathbb{F}$ is an ordered field and that $x \in \mathbb{F}$. Prove that $x \geq 0$ and $x \leq \varepsilon$ for all $\varepsilon > 0$ then $x = 0$.

2. Suppose that $\mathbb{F}$ is an ordered field and that $x, y \in \mathbb{F}$. Prove that $|xy| = |x| \cdot |y|$.

3. Suppose that $\mathbb{F}$ is an ordered field and that $x_1, \ldots, x_n \in \mathbb{F}$. Prove that

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$$

4. Let S be a nonempty bounded subset of $\mathbb{R}$ and let $m = \sup(S)$. Prove that $m \in S$ if and only if $m = \max(S)$.

5. Let S be a nonempty bounded subset of $\mathbb{R}$. Prove that $\sup(S)$ is unique.

6. Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Prove that $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$.

7. Let $x \in \mathbb{R}$. Prove that there is a unique $n \in \mathbb{N}$ such that $n - 1 \leq x < n$.

8. Let $x \in \mathbb{R}$. Prove that $x = \sup(q \in \mathbb{Q} \mid q < x)$.