LECTURE 13

# The Topology of the Reals

## 1. Neighborhoods

We have already used the well-ordered property of the reals to quantify the notion of distance: the distance between two points $x$ and $y$ on the real line is the absolute value of their difference. However, in what follows it is also useful to speak of points being sufficiently close to one another, as measured by some parameter, say $\varepsilon$.

DEFINITION 13.1. *Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A **neighborhood** (or $\varepsilon$-**neighborhood**) of $x$ is a set of the form*

$$N(x;\varepsilon) = \{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$$

DEFINITION 13.2. *Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A **deleted neighborhood** of $x$ is a set of the form*

$$N(x;\varepsilon) = \{y \in \mathbb{R} \mid |x - y| < \varepsilon \quad , \quad y \neq x\}$$

DEFINITION 13.3. *Let $S$ be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to lie in the **interior** of $S$ if there exists a neighborhood $N$ of $x$ such that $N \subseteq S$.*

In other words, a point $x$ is an interior point if one can find a small open interval about $x$ that completely lies in the set $S$.

DEFINITION 13.4. *Let $S$ be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is said to lie on the **boundary** of $S$ if for every neighborhood $N$ of $x$, $N \cap S \neq \phi$ and $N \cap (\mathbb{R}/S) \neq \phi$.*

In other words, a point $x$ lies on the boundary $S$ if every neighborhood $N$ of $x$ contains some points in $S$ and some points not in $S$. Note that a boundary point of $S$ does not necessarily have to be a point of $S$.

NOTATION 13.5. If $S$ is a subset of $\mathbb{R}$ we shall denote the set of its interior points as $int(S)$ and the set of its boundar points as $bnd(S)$.

EXAMPLE 13.6. Consider the set $S = (2, 3]$. The interior points of $S$ are the real numbers strictly greater than 2 and strictly less than 3. The boundary points of $S$ are 2 and 3. Note that the boundary point 2 does not lie in $S$ while the boundary point 3 does lie in $S$.

EXAMPLE 13.7. Consider the set $(1, 2) \cup (2, 3)$. The boundary points are 1, 2, and 3.

## 2. Closed and Open Sets

We have seen that a subset of $\mathbb{R}$ may contain all of its boundary points, some of its boundary points, or none of its boundary points. The first and last cases are of particular importance and so we introduce the following definitions.

DEFINITION 13.8. *Let $S \subseteq \mathbb{R}$. If the boundary of $S$ is contained in $S$ then we say that $S$ is a **closed** subset of $\mathbb{R}$. If the boundary of $S$ is contained in $\mathbb{R}/S$ then we say that $S$ is an **open** subset of $\mathbb{R}$.*

Although it may seem that the notions of open and closed are mutually exclusive; this is not so. For example, the set $\mathbb{R}$ itself is both open and closed. For $\mathbb{R}$ has no boundary, so its boundary set is $\{\}$; and of course $\{\} \subseteq S$ and $\{\} \subseteq \mathbb{R}/S$. Similarly, the empty set $\{\}$ has no boundary so it is both open and closed.

EXAMPLE 13.9. If $S = \{x_1, \ldots, x_n\}$ is a set containing only a finite number of points of $\mathbb{R}$ then $S$ is closed.

THEOREM 13.10. *Let $S$ be a subset of $\mathbb{R}$. Then*

(1) *$S$ is open if and only if $S = int(S)$*
(2) *$S$ is closed if and only if its complement $\mathbb{R}/S$ is open.*

THEOREM 13.11. *The union of any collection of open sets is an open set.*

*Proof.* Let $\mathcal{A}$ be an arbitrary collection of open sets and let

$$S = \bigcup_{A \in \mathcal{A}} A$$

If $x \in S$, then $x \in A$ for some $A \in \mathcal{A}$. Since $A$ is open, $x$ is an interior point of $A$. That is to say, there exists some neighborhood $N = N(x, \varepsilon)$ such that $N \subseteq A$. But $A \subseteq S$, so $N \subset S$, hence $x$ is an interior point of $S$. Since every point of $x \in S$ is an interior point, $S$ is open.

COROLLARY 13.12. *The intersection of any collection of closed sets is closed.*

*Proof.* This follows from the identity

$$\bigcap_{A \in \mathcal{A}} \mathbb{R}/A = \mathbb{R}/\left(\bigcup_{A \in \mathcal{A}} \mathbb{R}/A\right)$$

and Statement (2) of Theorem 13.9.

THEOREM 13.13. *The intersection of any finite collection of open sets is an open set.*

*Proof.* Let $A_1, \ldots, A_n$ be a finite collection of open sets and let

$$T = \bigcap_{i=1}^{n} A_i$$

If $T = \phi$ then we are done (the empty set is open). If $T \neq \phi$, then let $x$ be any element of $T$. Since $x$ lies in the intersection of the open sets $A_i$ it must be a member of each of them. Since each of the sets $A_i$ is open, $x$ must be an interior point of each of these sets. So within each of these sets $A_i$ there is a neighborhodd $N_i = N(x, \varepsilon_i)$ completely contained in $A_i$. Noting that

$$\varepsilon < \varepsilon' \qquad \Rightarrow \qquad N(x, \varepsilon) \subset N(x, \varepsilon')$$

we see that if we set

$$\varepsilon_{\min} = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$$

then

$$x \in N(x, \varepsilon_{\min}) \subseteq T$$

Thus, $x$ is an interior point of $T$ and hence $T$ is open.

COROLLARY 13.14. *The union of any finite collection of closed sets is closed.*

*Proof.* This follows from the identity

$$\bigcup_{A \in \mathcal{A}} \mathbb{R}/A = \mathbb{R}/\left(\bigcap_{A \in \mathcal{A}} \mathbb{R}/A\right)$$

and Statement (2) of Theorem 13.9.

EXAMPLE 13.15. For each $n \in \mathbb{N}$ set

$$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$$

Then each $A_n$ is an open set but

$$\bigcap_{i\in\mathbb{N}} A_i = \{0\}$$

which is closed.

## 3. Accumulation Points

Definition 13.16. *Let $S$ be a subset of $\mathbb{R}$. A point $x \in \mathbb{R}$ is an* ***accumulation point*** *of $S$ if every deleted neighborhood of $x$ contains a point in $S$.*

Example 13.17. Let $S = \left\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \dots, \frac{1}{n}, -\frac{1}{n}, \dots\right\}$. Then 0 is a accumulation point of $S$, but $0 \notin S$.

Remark 13.18. The notion of an accumulation point is very close to that of a boundary point, except that it is a bit more specific. For example, in the preceding example, the boundary points of $S$ are all the points $\pm\frac{1}{2}, \pm\frac{1}{3}, \dots$ as well as 0. However, 0 is the only accumulation point. And even if we included 0 in $S$ it would be an accumulation point.

Definition 13.19. *The* ***closure*** *of a subset $S$ of $\mathbb{R}$ is the union of $S$ and the set of all its accumulation points. We denote the closure of a subset $S$ of $\mathbb{R}$ by $cl(S)$.*

Here is a another characterization of the closure of a set $S$.

Proposition 13.20. *A point $x \in \mathbb{R}$ is in the closure of a subset $S \subset \mathbb{R}$ if and only if every neighborhood of $x$ intersects $S$.*

*Proof.*

$\Rightarrow$ Suppose $x \in cl(S)$. Then either $x \in S$ or $x$ is an accumulation point of $S$.

If $x \in S$, then for all $\varepsilon > 0$, $\{x\} \subseteq N(x, \varepsilon)$; hence $N(x,\varepsilon) \cap S \neq \{\}$.

If $x \in S'$, then by definition every neighborhood of $x$ must contain elements of $S$ and so again $N(x,\varepsilon) \cap S \neq \{\}$.

$\Leftarrow$ Suppose that for all $\varepsilon > 0$, $N(x,\varepsilon) \cap S \neq \{\}$. If $x \in S$, then certainly $x \in cl(S)$ and we're done. If $x \notin S$. Then we are assured that $N(x,\varepsilon) \cap S \neq \{x\}$ and so removing the point $x$ from $N(x,\varepsilon)$ does not affect the hypothesis that its intersection with $S$ is nonempty. Hence, every deleted neighborhood of $x$ will contain a point of $S$, so $x \in S' \subseteq cl(S)$.

Theorem 13.21. *Let $S$ be a subset of $\mathbb{R}$. Then*

(a) *$S$ is closed if and only if it contains all of its accumulation points.*
(b) *The closure of $S$ is a closed set.*
(c) *$S$ is closed if and only if $S = cl(S)$.*

*Proof.*

(a) $\Rightarrow$ Suppose that $S$ is closed and let $x$ is an an accumlation point of $S$. We must show $x \in S$. If $x \notin S$, then $x$ is in the open set $\mathbb{R}\backslash S$. Thus there exists a neighborhood $N$ of $x$ that lies completely in $\mathbb{R}\backslash S$. But then $N \cap S = \{\}$, so $x$ can not be an accumumation point if $x \notin S$.

$\Leftarrow$ Suppose $S$ contains all of its accumulation points. Let $x \in \mathbb{R}\backslash S$. If $x$ is not an accumulation point of $S$, then there exists a deleted neighborhood $N^*(x,\varepsilon)$such that $N^*(x,\varepsilon) \cap S = \{\}$. Since $x \notin S$, the whole neighborhood $N(x,\varepsilon) = N^*(x,\varepsilon) \cup \{x\}$ is outside of $S$. Thus, if $S$ contains all of its accumulation points, $\mathbb{R}\backslash S$ is open in $\mathbb{R}$. Since $S$ is then the complement of an open set, $S$ must be closed.

(b) By part (a) it suffices to show that the closure of $S$ contains all of its accumulation points. Thus suppose $x$ is an accumulation point of $S$. Then every deleted neighborhood of $x$ intersects $cl(S)$. We must show that $N^*(x,\varepsilon)$ intersects $S$. To this end, let $y \in N^*(x,\varepsilon) \cap cl(S)$. Since $N^*(x,\varepsilon)$ is an open set there exists a neighborhood $N(y,\delta)$ contained in $N^*(x,\varepsilon)$. But $y \in cl(S)$ so every neighborhood of $y$ intersects $S$. That is there exists a point $z \in N(y,\delta) \cap S$. But then $z \in N(y,\delta) \subseteq N^*(x,\varepsilon)$, so $x$ is an accumulation point of $S$ and $x \in cl(S)$.

(c) $\Rightarrow$ Suppose $S$ is closed but $S \neq cl(S) \equiv S \cup S'$, where $S'$ is the set of accumulation points of $S$. Hence there must be an accumulation $x$ point of $S$ that does not lie in $S$. But this contradicts statement (a) proven above. Hence if $S$ is closed then $S = cl(S)$.

$\Leftarrow$ Suppose $S = cl(S)$. Let $y$ be an arbitrary element of $\mathbb{R} \backslash S$. Then $y$ is neither in $S$ or an accumulation point of $S$. Hence it is to find a neighborhood $N(y,\varepsilon)$ that lies completely in $\mathbb{R}\backslash S$. Hence, $\mathbb{R}\backslash S$ is open. So $S$ is closed.

The definitions of an accumulation point is very close that of a boundary point, however the two concepts are not the same. To see this, we note:

**Proposition 13.22.** *If $x \in bd(S)$ and $x \notin S'$ (the set of accumulation points of $S$), then $x \in S$.*

*Proof*. Suppose the hypothesis is true, then since $x$ is not an accumulation there must exist a deleted neighborhood $N^*(x,\varepsilon)$ that does not contain any element of $S$. But since $x$ is also a boundary point of $S$, every neighborhood of $x$ must contain at least one point in $S$, in particular the neighborhood $N(x,\varepsilon) = N^*(x,\varepsilon) \cup \{x\}$. Since no point of $N^*(x,\varepsilon)$ lies in $S$, we can conclude $x \in S$.

We thus arrive at the following *picture* of an arbitrary subset $S$ of the real line.

$\mathbb{R}$ is the disjoint union of the following three sets:

(1) $int(S)$: the interior points of $S$
(2) $bd(S)$: the boundary points of $S$
(3) $ext(S)$ : the exterior point of $S$, $\mathbb{R}/S - bd(S)$

The exterior points of $S$ are never accumulation points. The interior points of $S$ are always accumulation points. The boundary points of $S$ may or may not be accumulation points. If a boundary point is an accumulation point, then there are infinitely many elements of $S$ that are arbitrarily close to it. If boundary point is not an an accumulation point, then, by virtue of the preceding proposition, it must be an element of $S$. We think of such points as the *isolated (boundary) points of $S$.*