LECTURE 14

Compact Sets

DEFINITION 14.1. *A **open covering** of a set S is a (possibly infinite) family $\mathcal{F}$ of open subsets of S such such that*

$$S \subseteq \bigcup_{T \in \mathcal{F}} T$$

DEFINITION 14.2. *A set S is said to be **compact** if every open covering of S contains a finite subcover.*

EXAMPLE 14.3. For each $n \in \mathbb{N}$, let $T_n = \left(\frac{1}{n}, 3\right)$ and set $\mathcal{F} = \{T_n \mid n \in \mathbb{N}\}$. Let $x \in S \equiv (0, 2)$. By the Archimedian property of $\mathbb{R}$, there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$. Thus, every $x \in S$ belongs to some T_n. Since the T_n are all open sets and every point of S lies in some T_n $\mathcal{F}$ is an open cover of S.

Now let $\mathcal{F}_1 = \{T_{n_1}, T_{n_2}, \dots, T_{n_r}\}$ be a finite subcollection of $\mathcal{F}$. It is easy to show that $\mathcal{F}_1$ cannot be an open cover of S. To see this, set

$$m = \max\{n_1, n_2, \dots, n_r\}$$

Then $T_m \in \mathcal{F}_1$ and because

$$n \leq m \quad \Rightarrow \quad (\frac{1}{n}, 3) \subseteq \left(\frac{1}{m}, 3\right)$$

We see that

$$T_m = \bigcup_{i=1}^{r} T_{n_i}$$

But $T_m \neq S$. For if $x > m$, then

$$0 < \frac{1}{x} < \frac{1}{m} < 3 \quad \Rightarrow \quad \left\{ \begin{array}{c} \frac{1}{x} \in S \\ \frac{1}{x} \notin \bigcup_{i=1}^{r} T_{n_i} = T_m \end{array} \right.$$

Hence, $S = (0, 2)$ is not compact (there exists an open cover of S such that there exists no finite subcover).

EXAMPLE 14.4. Suppose we modify the preceding example by setting $S' = [0, 2)$, and $\tilde{\mathcal{F}} = \left\{\tilde{T}_n = (-\frac{1}{n}, 3)\right\}$. Then, every individual $\tilde{T}_n$ is an open cover of S'. However, this does not mean that S is compact. For S' to be compact **every** open cover of S' must have a finite subcover. It is easy to see that the family of open subsets $\mathcal{F}' = \left\{T'_n = \left(-1, 2 - \frac{1}{n}\right)\right\}$ also covers S, but, in a manner similar to that of the preceding example, one cannot find a finite subcover of $\mathcal{F}'$ that covers S. So S' is not compact.

EXAMPLE 14.5. Finally, let us set $S'' = [0, 2]$. The familys $\mathcal{F}$ and $\mathcal{F}'$ now fail to cover S''; for

$$0 \notin \bigcup_{T \in \mathcal{F}} T$$

and

$$2 \notin \bigcup_{T' \in \mathcal{F}'} T'$$

Indeed, it turns out that there are no coverings S'' that provide counterexamples to the compactness condition (This will be seen to be a consequence of the Heine-Borel theorem, stated below).

By now you might have guessed that compact sets should be something akin to closed subsets of $\mathbb{R}$. However, remember that $\mathbb{R}$ itself is a closed subset of itself (since it contains its boundary, which is the empty set). But clearly $\mathbb{R}$ can not be compact (consider the open cover $\mathcal{F} = \{(-n, n)\}$ and why it cannot possess a finite subcover). It will turn out that the correct characterization of compact subsets of $\mathbb{R}$ is that they are closed subsets of $\mathbb{R}$ that are also bounded.

Lemma 14.6. *If S is a closed bounded subset of $\mathbb{R}$, then S has a minimal element and S has a maximal element.*

Proof. Since S is bounded from above, $m = \sup(S)$ exists. If $m \notin S$, then for each $\varepsilon > 0$ there exists an $x \in S$ such that $m - \varepsilon < x < m$. Thus, every deleted neighborhood $N^*(x, \varepsilon)$ will contain a element of S; so m is an accumulation point of S. But since S is closed, S contains all of its accumulation points. Hence, $m \equiv \sup(S) \in S$, so m is the maximum of S. A similar argument shows that $\inf(S) \in S$, so that S also possesses a minimum.

Theorem 14.7 (Heine-Borel). *A subset S of $\mathbb{R}$ is compact if and only if S is closed and bounded.*

Proof. (see text).

Theorem 14.8 (Bolzano-Weierstrass). *If S is subset S of $\mathbb{R}$ that is bounded and contains infinitely many points, then there exists at least one point in $\mathbb{R}$ that is an accumulation point of S.*

Proof. Let S be a bounded subset of $\mathbb{R}$ containing infinitely many points. Suppose S has no accumulation points. Then S is closed by Theorem 13.17(a). So by the Heine-Borel theorem S is compact. Since S has no accumulation points, given any $x \in S$, there exists a neighborhood $N(x, \varepsilon)$ such that $S \cap N(x, \varepsilon) = \{x\}$. The family

$$\mathcal{F} = \{N(x, \varepsilon) \mid x \in S\}$$

is evidently an open cover of S. (Here, for each $x \in S$, an ε is chosen so that $N(x, \varepsilon) \cap S = \{x\}$). Since S is compact, $\mathcal{F}$ has a finite subcover; that is to say, there exists points $x_1, x_2, \ldots, x_n \in S$ such that

$$\mathcal{F}' = \{N(x_1, \varepsilon_1), N(x_2, \varepsilon_2), \ldots, N(x_n, \varepsilon_n)\}$$

is an open cover of S. But now

$$S = S \cap \left(\bigcup_{i=1}^{n} N(x_i, \varepsilon_i) \right) = \{x_1, x_2, \ldots, x_n\}$$

is finite. This contradicts the hypothesis that S has infinitely many points. Hence, S must have an accumulation point.

Theorem 14.9. *Let $\mathcal{F} = \{K_\alpha : \alpha \in \mathcal{A}\}$ be a family of compact subsets of $\mathbb{R}$. Suppose that the intersection of any finite subfamily is non-empty. Then*

$$\bigcap_{\alpha \in \mathcal{A}} K_\alpha \neq \{\}$$

Proof. To prove the theorem, it suffices to show that there are points that lie in every member K_α of the family $\mathcal{F}$. Suppose we have a member K_1 of $\mathcal{F}$ such that no point of K_1 belongs to every K_α. (If we can not find such a K_1, then every subset $K_{\alpha'}$ will contain points that belong to every other K_α and the conclusion will follow.) Then every point of K_1 will belong to some $\mathbb{R} \backslash K_\alpha$, and so the sets F_α, $\alpha \in \mathcal{A}$ will form an open cover of K_1. Since K_1 is compact, there exist finitely many indices $\alpha_1, \ldots, a_n$ such that

$$K_1 \subseteq \bigcup_{i=1}^{n} F_{\alpha_i} = \bigcup_{i=1}^{n} \mathbb{R} \backslash K_{\alpha_i} = \mathbb{R} \backslash \left(\bigcap_{i=1}^{n} K_{\alpha_i} \right)$$

Thus, K_1 lives in the complement of $\bigcap_{i=1}^{n} K_{\alpha_i}$ and so

$$\{\} = K_1 \cap \left(\bigcap_{i=1}^{n} K_{\alpha_i} \right) = K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}$$

But, by hypothesis the intersection of any finite subcollection of the sets K_α is non-empty. Thus, some point in K_1 belongs to each K_α, and so

$$\bigcap_{\alpha \in \mathcal{A}} K_\alpha \neq \{\}$$

Corollary 14.10 (Nested Intervals Theorem). *Let $\mathcal{F} = \{A_n : n \in \mathbb{N}\}$ be a family of closed bounded intervals in $\mathbb{R}$ such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Then*

$$\bigcap_{n \in \mathbb{N}} A_n \neq \{\}$$

Proof. Given $n_1 < n_2 < \cdots < n_k$ in $\mathbb{N}$, we have

$$\bigcap_{i=1}^{k} A_i = A_{n_k} \neq \{\}$$

Thus, the hypotheses of the preceding theorem are satisfied and so

$$\bigcap_{n \in \mathbb{N}} A_n \neq \{\}$$