LECTURE 15

Sequences

DEFINITION 15.1. *A **sequence** of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$.*

Another way of thinking about a sequence is as a denumerable list of element of $\mathbb{R}$. Indeed, if $s : \mathbb{N} \to \mathbb{R}$ is a sequence (as defined above) setting

$$\begin{aligned} s_1 &= s(1) \\ s_2 &= s(2) \\ s_3 &= s(3) \\ &\vdots \end{aligned}$$

We can identify the function s with the ordered list of its values

$$\{s_1, s_2, s_3, \dots\}$$

We shall make use of this correspondence so often

$$s : \mathbb{N} \to \mathbb{R} \qquad \Longleftrightarrow \qquad \{s_1, s_2, s_3, \dots\}$$

that we shall employ a special notation that makes this identification implicit; denoting a sequence $s : \mathbb{N} \to \mathbb{R}$ by (s_n)

DEFINITION 15.2. *A sequence (s_n) of real numbers is said to **converge** to the real number S provided that for each $\epsilon > 0$, there exists a number $N \in \mathbb{Z}$ such that*

$$n > N \quad \Rightarrow \quad |s_n - S| < \epsilon \quad .$$

If a sequence (s_n) converges to S we will write

$$\lim_{n\to\infty} s_n = S$$

or

$$s_n \to S \quad .$$

*The number S is called the **limit** of the sequence (s_n). If a sequence does not converge to any real number, then the sequence is said to **diverge**.*

DEFINITION 15.3. *A sequence (s_n) in $\mathbb{R}$ is said to be **bounded** if there exists an $M \in \mathbb{R}$ such that*

$$|s_n| \leq M$$

THEOREM 15.4. *Convergent sequences are bounded.*

Proof. Let $(s_n)_{n\geq m}$ be a convergent sequence and let $s = \lim s_n$. According to the definition of the limit of a sequence, for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad |s_n - s| < \epsilon \quad .$$

Adding $|s|$ to both sides of the inequality on the right we get

$$|s_n - s| + |s| < \epsilon + |s| \quad . \tag{15.1}$$

By the Triangle Inequality, we have

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| \quad , \tag{15.2}$$

and so combining the inequalities (15.1) and (15.2) we get

$$|s_n| < \epsilon + |s| \quad . \tag{15.3}$$

We can in particular choose $\epsilon = 1$ and find an $N \in \mathbb{N}$ such that

$$|s_n| < |s| + 1 \quad \text{for all } n > N \ . \tag{15.4}$$

Thus, the set of terms $\{s_n \mid n > N\}$ is bounded from above. Set

$$M = max\,\{|s| + 1, s_1, s_2, \ldots, s_N\}$$

then we have

$$s_n \leq M \quad \text{for all } n \geq m \quad ,$$

and so the sequence is bounded. □

THEOREM 15.5. *If a sequence (s_n) has a limit S, then that limit is unique.*

Proof.

Suppose

$$\lim_{n\to\infty} s_n = S$$

and also

$$\lim_{n\to\infty} s_n = T.$$

That is to say, suppose for every $\epsilon > 0$, there is an $N_1 \in \mathbb{N}$ such that

$$N_1 < n \quad \Rightarrow \quad |s_n - S| < \frac{\epsilon}{2}$$

and for every $\epsilon > 0$, there is an $N_2 \in \mathbb{N}$ such that

$$N_2 < n \quad \Rightarrow \quad |s_n - T| < \frac{\epsilon}{2} \quad .$$

For any $n > max\,\{N_1, N_2\}$ we must have

$$|T - S| = |(s_n - S) - (s_n - T)| \leq |s_n - S| + |s_n - T| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \quad .$$

Thus,

$$0 \leq |T - S| < \epsilon \quad \text{for all } \epsilon > 0.$$

Thus, $|T - S| = 0$, hence $T = S$. □