Limit Theorems for Sequences

THEOREM 16.1. *Suppose that* (s_n) *and* (t_n) *are convergent sequences with* $\lim s_n = s$ *and* $\lim t_n = t$. *Then*

(1) $lim(s_n + t_n) = s + t$
(2) $lim(ks_n) = ks$ *for all* $k \in \mathbb{R}$.
(3) $lim(s_n t_n) = st$
(4) $lim(s_n/t_n) = s/t$, *provided that* $t_n \neq 0$ *for all* n *and* $t \neq 0$.

Proof.

(a) To show that $\lim(s_n + t_n) = s + t$ we need to demonstrate that we can force $(s_n + t_n) - (s + t)$ to be as small as we like by choosing n to be sufficiently large. By the Triangle Inequality we have

$$|(s_n + t_n) - (s + t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t|$$

Now let $\varepsilon > 0$. Since (s_n) is convergent, there exists a number N_1 such that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \text{for all } n > N_1$$

Similarly, since (t_n) is convergent there exists a number N_2 such that

$$|t_n - t| < \frac{\varepsilon}{2} \quad \text{for all } n > N_2$$

Set $N = \max\{N_1, N_2\}$ so that

$$\left(|s_n - s| < \frac{\varepsilon}{2} \quad \text{and} \quad |t_n - t| < \frac{\varepsilon}{2}\right) \qquad \text{for all } n > N$$

We now have

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon \qquad \text{for all } n > N$$

So the sequence $(s_n + t_n)$ converges to $s + t$. □

(b) We assume $k \neq 0$ since the result is trivial for $k = 0$. We need to show that for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$|ks_n - ks| < \epsilon \quad \text{for all } n > N \ .$$

Since (s_n) converges, we know that for any $\epsilon' > 0$ there is an $N' \in \mathbb{N}$ such that

$$|s_n - s| \leq \epsilon' \quad \text{for all} n > N' \quad .$$

In particular, we can choose $\epsilon' = \frac{\epsilon}{|k|}$ and find an N' such that

$$|s_n - s| \leq \frac{\epsilon}{|k|} \quad \text{for all} n > N' \quad .$$

Multiplying both sides of the inequality on the left by the positive number $|k|$ we get

$$|k||s_n - s| = |ks_n - ks| < \epsilon \quad .$$

So setting $N = N'$ we get the statement we need

$$|ks_n - ks| < \epsilon \quad \text{for all } n > N \ .$$

□

(c) Since every convergent sequence is bounded, there exists an $M_1 \in \mathbb{R}$ such that $|s_n| \leq M_1$ for all $n \in \mathbb{N}$. Set $M = \max\{M_1, |t|\}$. We then have

$$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n t_n - s_n t| + |s_n t - st| \leq M\,|t_n - t| + M\,|s_n - s|$$

so

$$|s_n t_n - st| \leq M\,|t_n - t| + M\,|s_n - s|$$

Now let $\varepsilon > 0$. Since (s_n) and (t_n) are convergent sequences, there exist numbers N_1 and N_2 such that

$$\begin{aligned} |t_n - t| &< \frac{\varepsilon}{2M} \quad \text{for all } n > N_1 \\ |s_n - s| &< \frac{\varepsilon}{2M} \quad \text{for all } n > N_2 \end{aligned}$$

Let $N = \max\{N_1, N_2\}$. Then we have for all $n > N$

$$|s_n t_n - st| \leq M\,|t_n - t| + M\,|s_n - s| < M\left(\frac{\varepsilon}{2M}\right) + M\left(\frac{\varepsilon}{2M}\right) = \varepsilon$$

So $\lim(s_n t_n) = st$. □

(d) Since $s_n/t_n = s_n * (1/t_n)$, it suffices from Part (c) to show that $\lim(1/t_n) = 1/t$. Let $\varepsilon > 0$. We need to show that by choosing $n \in \mathbb{N}$ sufficiently large that we can force

$$\left|\frac{1}{t_n} - t\right| < \varepsilon$$

Now

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t - t_n}{t_n t}\right|$$

To get a lower bound on how small the denominator might be we note that since (t_n) converges there exists a number N_1 such that

$$n > N_1 \quad \Rightarrow \quad |t_n - t| < \frac{|t|}{2}$$

Thus, for $n > N_1$ we have

$$|t_n| = |t - (t - t_n)| \geq |t| + |t - t_n| > |t| - \frac{|t|}{2} = \frac{|t|}{2}$$

Thus, we have

$$\frac{1}{|t_n t|} = \frac{1}{|t|\,|t_n|} < \frac{1}{|t|\,\left|\frac{t}{2}\right|} = \frac{2}{|t|^2}$$

As for the numerator, there must exist an $N_2 \in \mathbb{N}$ such that

$$n > N_2 \quad \Rightarrow \quad |t_n - t| < \frac{|t|^2}{2}\varepsilon$$

And so for $n > N_2$ and $n > N_2$ we'll have

$$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t - t_n}{t_n t}\right| = (|t - t_n|)\,\frac{1}{|t_n t|} < \left(\frac{|t|^2}{2}\varepsilon\right)\left(\frac{2}{|t|^2}\right) = \varepsilon$$

and so $\lim(1/t_n) = 1\backslash t$. □

We can strengthen the statements of the theorems presented in the text by relaxing the hypotheses about something being true for all n to the requirement that they are true for sufficiently large n. To make this idea precise, we'll utilize the following definition.

Definition 16.2. *We'll say that a property of a sequence (s_n) holds* ***for all sufficiently large*** *n if there exists an integer $N_0 \in \mathbb{N}$ such that the property is true whenever $n > N_0$*

Theorem 16.3. *Suppose that (s_n) and (t_n) are convergent sequences with $\lim s_n = s$ and $\lim t_n = t$. If $s_n \leq t_n$ for all sufficiently large n then $s \leq t$.*

Proof. Suppose $s > t$ then

$$\varepsilon = \frac{s-t}{2} > 0$$

Thus there exists an $N_1 \in \mathbb{N}$ such that

$$n > N_1 \quad \Rightarrow \quad s - \varepsilon < s_n < s + \varepsilon$$

Similarly, there exists an $N_2 \in \mathbb{N}$ such that

$$n > N_2 \quad \Rightarrow \quad t - \varepsilon < t_n < t + \varepsilon$$

Let $N = \max\{N_0, N_1, N_2\}$. Then we have simultaneously for all $n > N$

$$\begin{aligned} s_n &\leq t_n \\ t_n &< t + \varepsilon = \frac{s+t}{2} \\ \frac{s+t}{2} &= s - \varepsilon < s_n \end{aligned}$$

However, the last two inequalities imply that $t_n < s_n$ which contradicts the first inequality. and so we must have $s \leq t$. □

COROLLARY 16.4. *If (s_n) converges to s and $s_n \geq 0$, and $s_n \geq 0$ for all sufficiently large n, then $s \geq 0$..*

COROLLARY 16.5 (Squeeze Theorem). *If (a_n) and (c_n) are convergent sequences such that*

$$\lim a_n = a \quad , \quad \lim c_n = a$$

and (b_n) is a sequence such that for all sufficiently large n

$$a_n \leq b_n \leq c_n$$

then

$$\lim b_n = a$$

LEMMA 16.6. *Suppose c is a number such that $0 < c < 1$. Then*

$$\lim (c^n) = 0$$

Proof. We need to show that for any $\varepsilon > 0$ there exists a natural number N such that

$$n > N \quad \Rightarrow \quad c^n = |c^n - 0| < \varepsilon \tag{16.1}$$

Now if $0 < c < 1$ then there exists a $y > 0$ such that

$$\frac{1}{1+y} = c$$

so

$$c^n = \frac{1}{(1+y)^n} = \frac{1}{1 + ny + \cdots + ny^{n-1} + y^n} = \frac{1}{ny + (1 + \cdots + ny^{n-1} + y^n)} < \frac{1}{ny}$$

Now, by the Archimedian Property of the reals we can make $\frac{1}{ny}$ as small as we like by choosing n to be sufficiently large. Indeed, so long as $\frac{1}{n} < y\varepsilon$ we'll have

$$c^n < \frac{1}{ny} < \varepsilon$$

and so (16.1) is true for $N = 1/(y\varepsilon) = \frac{c}{(1-c)\varepsilon}$. □

COROLLARY 16.7. *Suppose $|c| < 1$ and M is any fixed real number. Then*

$$\lim (Mc^n) = 0$$

THEOREM 16.8. *Suppose that (s_n) is a sequence of positive terms and that*

$$L = \lim (s_{n+1}/s_n)$$

exists. Then if $L < 1$ then $\lim s_n = 0$.

Proof. Assume $L < 1$, by a corollary above we have also (since each term $s_{n+1}/s_n > 0$), and there exists a real number c such that $L < c < 1$. Let $\varepsilon = c - L$ so that $\varepsilon > 0$. Then there exists a natural number N such that

$$n > N \quad \Rightarrow \quad \left|\frac{s_{n+1}}{s_n} - L\right| < \varepsilon$$

but then for $n > N$

$$\frac{s_{n+1}}{s_n} = \left|\frac{s_{n+1}}{s_n}\right| = \left|L + \frac{s_{n+1}}{s_n} - L\right| \leq |L| + \left|\frac{s_{n+1}}{s_n} - L\right| < L + \varepsilon = L + (c - L) = c < 1$$

It follows that for all $n > N$

$$s_{n+1} < s_n c$$

so

$$\begin{aligned} s_{N+1} &< cs_N \\ &\Rightarrow \quad s_{N+2} < cs_{N+1} < c^2 s_N \\ &\Rightarrow \quad s_{N+3} < cs_{N+2} < c^3 s_N \\ &\vdots \\ &\Rightarrow \quad s_{N+k} < cs_{N+k-1} < c^k s_N \end{aligned}$$

Let

$$M = \frac{s_N}{c^N}$$

Then

$$0 < s_n < c^n M$$

for all $n > N$. By the preceding corollary and the Squeeze Theorem we can conclude

$$\lim s_n = 0$$

□

1. Infinite Limits

Consider the following two sequences

$$\begin{aligned} s_n &= \sin(n) \\ t_n &= (3)^n \end{aligned}$$

Of course, neither of these sequences converges, but each has a nice property. The first is a bounded sequence since every term has absolute value ≤ 1. Yet, this sequence behaves very erratically, its terms taking values all over the interval between -1 and 1. The second sequence is unbounded, but at least there is a consistent behavior for successive terms; each term is larger than the preceding term, and somewhat like the Archimedean Property of the Reals, for any $x \in \mathbb{R}$ we can always find an $n \in \mathbb{N}$ such that $t_n > x$. This section deals with formalizing properties of unbounded sequences.

Definition 16.9. *A sequence (s_n) is said to* ***diverge to*** $+\infty$ *if for every $M \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that*

$$n > N \quad \Rightarrow \quad s_n > M$$

Similarly, a sequence (s_n) is said to ***diverge to*** $-\infty$ *if for every $M \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that*

$$n > N \quad \Rightarrow \quad s_n < M$$

NOTATION 16.10. If a sequence (s_n) diverges to $+\infty$ we shall write

$$\lim s_n = +\infty$$

and if a sequence (s_n) diverges to $-\infty$ we shall write

$$\lim s_n = -\infty$$

THEOREM 16.11. *Suppose that (s_n) and (t_n) are sequences such that $s_n \leq t_n$ for all n sufficiently large. Then*

(1) *If $\lim s_n = +\infty$, then $\lim t_n = +\infty$.*
(2) *If $\lim t_n = -\infty$, then $\lim s_n = -\infty$.*

THEOREM 16.12. *Let (s_n) be a sequence of positive numbers. Then*

$$\lim s_n = +\infty \quad \Longleftrightarrow \quad \lim \left(\frac{1}{s_n}\right) = 0$$

Proof.

$\Rightarrow$ Suppose that $\lim s_n = +\infty$. Given $\varepsilon > 0$, let $M = 1/\varepsilon$. Since (s_n) diverges to $+\infty$, there exists an $N \in \mathbb{N}$ such that

$$n > N \quad \Rightarrow \quad s_n > M$$

But this means

$$n > N \quad \Rightarrow \quad \left|\frac{1}{s_n} - 0\right| = \frac{1}{s_n} < \frac{1}{M} = \varepsilon$$

so

$$\lim \left(\frac{1}{s_n}\right) = 0.$$

$\Leftarrow$ This is left as an exercise.

□