LECTURE 17

Monotone Sequences and Cauchy Sequences

Definition 17.1. *A sequence (s_n) of real numbers is* ***increasing*** *if $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$, and is* ***decreasing*** *if $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$. If a sequence is* ***monotone*** *if it is either increasing or decreasing.*

Theorem 17.2 (Monotone Convergence Theorem). *A monotone sequence is convergent if and only if it is bounded.*

Proof.

$\Rightarrow$ We have already proved (Theorem 16.13 in the text) that every convergent sequence is bounded. So, in particular, if (s_n) is a convergent monotone sequence then it is bounded.

$\Leftarrow$ Suppose (s_n) is a bounded increasing sequence. By the Completeness Axiom the set

$$S = \{s_1, s_2, s_3, \dots\}$$

must have a least upper bound since it is bounded. Let $s = \sup(S)$. We claim $\lim s_n = s$. To prove this we simply note that given any $\varepsilon > 0$, $s - \varepsilon$ is not an upper bound for S. Therefore, there exists an N such that $s_N > s - \varepsilon$. Furthermore, since (s_n) is increasing $s_n \geq s_N$ for all $n > N$. Hence,

$$n > N \quad \Rightarrow \quad s - \varepsilon < s_n \Rightarrow \quad |s_n - s| < \varepsilon$$

and so (s_n) converges to s.

The case when (s_n) is a decreasing monotone sequence is similar. □

Theorem 17.3. *Suppose (s_n) is a unbounded monotone sequence.*

(1) *If (s_n) is increasing then* $\lim s_n = +\infty$.
(2) *If (s_n) is decreasing then* $\lim s_n = -\infty$.

Proof.

(a) Let (s_n) be an increasing sequence and that $S = \{s_1, s_2, \dots\}$ is unbounded. Since (s_n) is increasing S will be bounded from below by s_1, so S must be unbounded from above. Thus, given any $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that $s_N > M$. But since (s_n) is increasing we also have

$$n > N \quad \Rightarrow \quad s_n \geq s_N \quad \Rightarrow \quad s_n > M .$$

So, by definition, $\lim s_n = +\infty$.

The proof of (b) is similar. □

Definition 17.4. *A sequence of real numbers is said to be a* ***Cauchy sequence*** *if for every $\varepsilon > 0$ there exists a number N such that $m, n > N$ implies that $|s_n - s_m| < \varepsilon$.*

Lemma 17.5. *Every convergent sequence is a Cauchy sequence.*

Proof. Suppose that (s_n) converges to s. To show that, for sufficiently large m and n, s_m is close to s_n, we use the fact that they are both close to s. From the Triangle Inequality we have

$$|s_n - s_m| = |s_n - s - (s_m - s)| \leq |s_n - s| + |s_m - s|$$

Now fix $\varepsilon > 0$. Since (s_n) converges to s there exists an N such that

$$k > N \quad \Rightarrow \quad |s_k - s| < \frac{\varepsilon}{2}$$

This then implies that

$$n, m > N \quad \Rightarrow \quad |s_n - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$$

so the sequence (s_n) is a Cauchy sequence.

LEMMA 17.6. *Every Cauchy sequence is bounded.*

Proof. (Homework problem)

THEOREM 17.7 (Cauchy Convergence Theorem). *A sequences of real numbers is convergent if and only if it is Cauchy sequence.*

*Proof.*We have already seen that every convergent sequence is a Cauchy sequence. We shall now prove the converse, that every Cauchy sequence converges.

Let (s_n) be a Cauchy sequence and suppose $S = \{s_1, s_2, \dots\}$ is the set of its values (without duplication). We consider two cases:

- Suppose S is finite. Let
$$\delta = \min\{|s_i - s_j| \mid s_i, s_j \in S\},$$
the minimal distance between distinct members of S. Note that $\delta > 0$. Since (s_n) is Cauchy, there exists a $N \in \mathbb{N}$ such that
$$m, n > N \quad \Rightarrow \quad |s_n - s_m| < \delta$$
But this cannot happen unless $s_n = s_m$ since the minimal distance between distinct s_n and s_m is δ. Thus,
$$m, n > N \quad \Rightarrow \quad s_n = s_m = s_{N+1}$$
and so the sequence (s_n) must converge to s_{N+1}.
- Suppose S is infinite. From the preceding lemma we know that S is bounded. The Bolzano-Weierstrass Theorem then tells us that S must have an accumulation point. Let s be such an accumulation point. We claim that (s_n) converges to s.
  Let $\varepsilon > 0$. Since (s_n) is Cauchy, there exists an N such that
$$n, m > N \quad \Rightarrow \quad |s_n - s_m| < \frac{\varepsilon}{2}$$
Since s is an accumulation point, the neighborhood
$$N(s, \varepsilon/2) = (s - \varepsilon/2, s + \varepsilon/2)$$
contains infinitely many points of S. Thus, in particular there must exist an $m > N$ such that $s_m \in N(s, \varepsilon/2)$. Hence for any $n > N$ we have
$$|s_n - s| = |s_n - s_m + s_m - s| \leq |s_n - s_m| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
Hence,
$$\lim(s_n) = s.$$

□