Math 4063-5023

SOLUTIONS TO SECOND EXAM

9:00 – 10:14 , November 10, 2015

1. Definitions. Write down the precise definitions of the following notions. (5 pts each)

(a) a **vector space homomorphism** (a.k.a. linear transformation)

- A mapping $T : V \to W$ between two vector spaces is a vector space homomorphism if

$$\begin{aligned} T(\lambda v) &= \lambda T(v) \qquad \text{for all } \lambda \in \mathbb{F} \text{ and all } v \in V \\ T(v + v') &= T(v) + T(v') \qquad \text{for all } v, v' \in V \end{aligned}$$

(b) the **kernel of a vector space homomorphism**

- If $T : V \to W$ is a vector space homomorphism then

$$\ker(T) = \{v \in V \mid T(v) = 0_W\}$$

(c) the **range of a vector space homomorphism**

- If $T : V \to W$ is a vector space homomorphism then

$$range(T) = \{w \in W \mid w = T(v) \text{ for some } v \in V\}$$

(d) a **vector space isomorphism**

- If $T : V \to W$ is a vector space isomorphism if

$$\begin{aligned} range(T) &= W \\ \ker(T) &= \{0_V\} \end{aligned}$$

(e) The **quotient space** V/S (where S is a subspace of a vector space V).

- Let

$$v + S = \{v + s \in V \mid s \in S\} \qquad \text{(the } S\text{-hyperplane through } v\text{)}$$

then

$$V/S = \{v + S \mid v \in V\} \qquad \text{(the set of } S\text{-hyperplanes in } V\text{)}$$

2. (15 pts) A function $f : V \longrightarrow W$ is injective if $f(v) = f(u)$ implies $v = u$. Show that a linear transformation $T : V \longrightarrow W$ is injective **if and only if** its kernel is $\{0_V\}$.

- $\Rightarrow$ We first note that for any linear transformation $T(0_V) = T(0_{\mathbb{F}} \cdot v) = 0_{\mathbb{F}} \cdot T(v) = 0_W$, and so 0_V always belongs to $\ker(T)$. Now suppose T is injective and $v \in \ker(T)$. We have

$$\begin{aligned} v \in \ker(T) &\Rightarrow T(v) = 0_W = T(0_V) \\ &\Rightarrow v = 0_V \text{ since } T \text{ is injective} \end{aligned}$$

Therefore, 0_V is the only vector in $\ker(T) : \ker(T) = \{0_V\}$.

$\Longleftarrow$ Suppose $\ker(T) = \{0_V\}$ and $T(v) = T(v')$. Then

$$\begin{aligned} 0_W &= T(v) - T(v') \\ &= T(v - v') \qquad \text{because } T \text{ is a linear transformation} \\ &\Rightarrow v - v' \in \ker(T) \qquad \text{by the definition of } \ker(T) \\ &\Rightarrow v - v' = 0_V \qquad \text{since by hypothesis } \ker(T) = \{0_V\} \\ &\Rightarrow v = v' \end{aligned}$$

Thus, if $\ker(T) = \{0_V\}$, then $T(v) = T(v') \quad \Rightarrow \quad v = v'$ and so T is injective.

3. Let $\mathcal{P}_n$ be the vector space of polynomials of degree $\leq n$. Consider the function $T : \mathcal{P}_3 \to \mathcal{P}_3$ given by

$$T(p) = \frac{d^2p}{dx^2} - 2x\frac{dp}{dx} + 2p$$

(a) (5 pts) Show that T is a linear transformation (Hint: use general properties of derivatives rather than an explicit basis for $\mathcal{P}_n$)

- We just need to show $T(\alpha p_1 + \beta p_2) = \alpha T(p_1) + \beta T(p_2)$ for any $\alpha, \beta \in \mathbb{R}$ and any $p_1, p_2 \in P_n$. We have

$$\begin{aligned} T(\alpha p_1 + \beta p_2) &= \frac{d^2}{dx^2}(\alpha p_1 + \beta p_2) - 2x\frac{d}{dx}(\alpha p_1 + \beta p_2) + 2(\alpha p_1 + \beta p_2) \\ &= \alpha\frac{d^2p_1}{dx^2} + \beta\frac{d^2p_2}{dx^2} - 2\alpha x\frac{dp_1}{dx} - 2\beta\frac{dp_2}{dx} + 2\alpha p_1 + 2\beta p_2 \\ &= \alpha\left(\frac{d^2p_1}{dx^2} - 2x\frac{dp_1}{dx} + 2p_1\right) + \beta\left(\frac{d^2p_2}{dx^2} - 2x\frac{dp_2}{dx} + 2p_2\right) \\ &= \alpha T(p_1) + \beta T(p_2) \qquad \checkmark \end{aligned}$$

(b) (5 pts) Find the matrix representing T acting on $\mathcal{P}_2$ (use the standard basis $\{1, x, x^2\}$ of $\mathcal{P}_2$).

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$$\begin{aligned} 1 &\longmapsto 0 - 2\cdot 0 + 2 * 1 = 2 = 2\cdot 1 + 0\cdot x + 0\cdot x^2 &\to\quad T(1)_B = [2,0,0] \\ x &\longmapsto 0 - 2\cdot x + 2x = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 &\to\quad T(x)_B = [0,0,0] \\ x^2 &\longmapsto 2 - 4x^2 + 2x^2 = 2\cdot 1 + 0\cdot x - 2\cdot x^2 &\to\quad T(x^2)_B = [2,0,-2] \end{aligned}$$

So

$$\mathbf{A}_{T,B} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ T(1)_B & T(x)_B & T(x^2)_B \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(c) (5 pts) Find a basis for the range of T (with basis vectors expressed as polynomials),

- $\mathbf{A}_{T,B}$ row reduces to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

From this we see that the first and last columns of $\mathbf{A}_{T,B}$ will provide a basis for $ColSp(\mathbf{A}_{T,B})$. But $\ker(T) \longleftrightarrow ColSp(\mathbf{A}_{T,B})$, so mapping $[2,0,0]$ and $[2,0,-2]$ back to polynomials, we get

$$Range(T) = span\left(2, 2 - 2x^2\right)$$

(d) (5 pts) Find a basis for the kernel of T (with basis vectors expressed as polynomials).

- $\ker(T)$ corresponds to the null space of $\mathbf{A}_{T,B}$. From the reduced row echelon form of $\mathbf{A}_{T,B}$ we see that the solution to $\mathbf{A}_{T,B}\mathbf{x} = \mathbf{0}$, are given by

$$\begin{matrix} x_1 = 0 \\ x_3 = 0 \\ 0 = 0 \end{matrix} \quad\Rightarrow\quad \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus, $[0,1,0]$ is a basis for the null space of $\mathbf{A}_{T,B}$. Mapping it back to a polynomial, we get

$$\ker(T) = span(x)$$

4. (10 pts) Let S be a subspace of a vector space V. Suppose $\{v_1,\ldots,v_k\}$ is a basis for V and $\{v_1,\ldots,v_k,v_{k+1},\ldots,v_n\}$ is a compatible basis for V. Let $p_S : V \to V/S : v \longmapsto v+S$ be the canonical projection. Show that $\{p_S(v_{k+1}),\ldots,p_S(v_n)\}$ is a basis for V/S.

- We need to show two things: (i) that $V/S = span(p_S(v_{k+1}),\ldots,p_S(v_n))$ and (ii) the vectors $p_S(v_{k+1}),\ldots,p_S(v_n)$ are linearly independent.

  (i) Since $p_S : V \to V/S$ is surjective, every element of V/S is of the form $p_S(v)$ for some vector $v \in V$. Thus

$$\begin{aligned}
V/S &= \{p_S(v) \mid v \in V\} \\
&= \{p_S(a_1v_1+\cdots a_kv_k + a_{k+1}v_{k+1}+\cdots+a_nv_n) \mid a_1,\ldots,a_n \in \mathbb{F}\} && \text{since the } v_i \text{ form a basis for } V \\
&= \{a_1p_S(v_1)+\cdots+a_kp_S(v_k)+a_kp_S(v_{k+1})+\cdots+a_np_S(v_n) \mid a_1,\ldots,a_n \in F\} && \text{since } p_S \text{ is a linear transformation} \\
&= \{0_V+\cdots+0_V+a_{k+1}p_S(v_{k+1}) \mid a_1,\ldots,a_n \in \mathbb{F}\} && \text{since } v_1,\ldots,v_k \in S = \ker(p_S) \\
&= span(p_S(v_{k+1}),\ldots,p_S(v_n)) \quad \checkmark
\end{aligned}$$

  (ii) Now we'll show that the vectors $p_S(v_{k+1}),\ldots,p_S(v_n) \in V$ are linearly independent. Suppose

$$(*) \qquad a_{k+1}p_S(v_{k+1})+\cdots+a_np_S(v_n) = 0_{V/S}$$

  Then, since p_S is linear transformation, this implies

$$p_S(a_{k+1}v_{k+1}+\cdots+a_nv_n) = 0_{V/S} \quad \Rightarrow \quad a_{k+1}v_{k+1}+\cdots+a_nv_n \in \ker(p_S) = S$$

  But the vectors $v_{k+1},\ldots,v_n$, by construction are linearly independent vectors outside of S. Therefore,

$$a_{k+1}v_{k+1}+\cdots+a_nv_n \in S \quad \Rightarrow \quad a_{k+1} = 0_{\mathbb{F}}\ ,\ldots\ ,a_n = 0_{\mathbb{F}}$$

  Hence, (*) requires $a_{k+1} = a_{k+2} = \cdots = a_n = 0_{\mathbb{R}}$, and so the vectors $p_S(v_{k+1}),\ldots,p_S(v_n)$ are linearly independent.

5.

(a) (5 pts) Compute the determinant of $\mathbf{A} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix}$ using a cofactor expansion.

- Cofactor expansion along the last row:

$$\begin{aligned}
\det(\mathbf{A}) &= 0+(1)(-1)^{3+2}\det\begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}+0 \\
&= -(2+2) \\
&= -4
\end{aligned}$$

b. (5 pts) Compute the determinant of $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$ using elementary row operations.

$$\begin{aligned}
\det(\mathbf{A}) &= \det\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_3} = -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\
\xrightarrow{R_3 \to R_3 - R_2} &= -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -(1)(1)(1) \\
&= -1
\end{aligned}$$

6.

(a) (5 pts) Compute the cofactor matrix $\mathbf{C}$ of $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{aligned} C_{11} &= (-1)^{1+1} \det\left(\mathbf{A}^{(1,1)}\right) = \det([d]) = d \\ C_{12} &= (-1)^{1+2} \det\left(\mathbf{A}^{(1,2)}\right) = -\det([c]) = -c \\ C_{21} &= (-1)^{2+1} \det\left(\mathbf{A}^{(2,1)}\right) = -\det([b]) = -b \\ C_{22} &= (-1)^{2+2} \det\left(\mathbf{A}^{(2,2)}\right) = \det([a]) = a \end{aligned}$$

$$\mathbf{C} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

(b) (5 pts) Use the result of (a) to compute $\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^T = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

7. (10 pts) Let $B_1 = \left\{1, x, x^2\right\}$ and let $B_2 = \left\{1, x-1, (x-1)^2\right\}$. Regarding B_1 and B_2 as bases for the vector space of polynomials of degree ≤ 2, find the change-of-coordinates-matrix that converts coordinate vectors with respect to B_1 to coordinate vectors with respect to B_2.

- The change of basis matrix $\mathbf{C}_{B_1 \to B_2}$ is formed by figuring out the coordinate vectors of the vectors in B_1 with respect to the basis B_2.

$$\begin{aligned} 1 &= 1 \cdot 1 + 0 \cdot (x-1) + 0 \cdot (x-1)^2 \quad \Rightarrow \quad [1]_{B_2} = [1,0,0] \\ x &= 1 \cdot 1 + 1 \cdot (x-1) + 0 \cdot (x-1)^2 \quad \Rightarrow \quad [x]_{B_2} = [1,1,0] \\ x^2 &= 1 \cdot 1 + 2 \cdot (x-1) + 1 \cdot (x-1)^2 \quad \Rightarrow \quad \left[x^2\right]_{B_2} = [1,2,1] \end{aligned}$$

So

$$\mathbf{C}_{B \to B'} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ [1]_{B_2} & [x]_{B_2} & \left[x^2\right]_{B_2} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$