MATH 4063-5023
Solutions to Homework Set 1

1. Let $\mathbb{F}$ be a field, and let $\mathbb{F}^n$ denote the set of n-tuples of elements of $\mathbb{F}$, with operations of scalar multiplication and vector addition defined by

$$\begin{aligned} \lambda \cdot [\alpha_1, \ldots, \alpha_n] \; &: \; = [\lambda a_1, \ldots, \lambda a_n] \quad , \quad \text{for all } \lambda \in \mathbb{F} \text{ and all } [\alpha_1, \ldots, \alpha_n] \text{ in } \mathbb{F}^n \\ [\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n] \; &: \; = [\alpha_1 + \beta_i, \ldots, \alpha_n + \beta_n] \quad , \quad \text{for all } [\alpha_1, \ldots, \alpha_n] \text{ and } [\beta_1, \ldots, \beta_n] \text{ in } \mathbb{F}^n \end{aligned}$$

Check that $\mathbb{F}^n$ satisfies all the axioms of a vector space over $\mathbb{F}$.

- There are 8 axioms to check. We'll check them one by one, constantly using the hypothesis that $\mathbb{F}$ is a field (and so obeys the 9 axioms of a field (see Definition 1.7 of Lecture 1).

(i) Commutativity of Vector Addition

$$\begin{aligned} [\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n] \; &: \; = [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n] \\ &= [\beta_1 + \alpha_1, \ldots, \beta_n + \alpha_n] \quad \text{because addition in } \mathbb{F} \text{ is commutative} \\ &: \; = [\beta_1, \ldots, \beta_n] + [\alpha_1, \ldots, \alpha_n] \end{aligned}$$

(ii) Associativity of Vector Addition

$$\begin{aligned} ([\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n]) + [\gamma_1, \ldots, \gamma_n] \; &: \; = [\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n] + [\gamma_1, \ldots, \gamma_n] \\ &: \; = [(\alpha_1 + \beta_1) + \gamma_1, \ldots, (\alpha_n + \beta_n) + \gamma_n] \\ &= [\alpha_1 + (\beta_1 + \gamma_1), \ldots, \alpha_n + (\beta_n + \gamma_n)] \quad \text{because addition in } \mathbb{F} \text{ is associative} \\ &: \; = [\alpha_1, \ldots, \alpha_n] + [\beta_1 + \gamma_1, \ldots, \beta_n + \gamma_n] \\ &: \; = [\alpha_1, \ldots, \alpha_n] + ([\beta_1, \ldots, \beta_n] + [\gamma_1, \ldots, \gamma_n]) \end{aligned}$$

(iii) Existence of Additivity Identity.
Set $\mathbf{0}_{\mathbb{F}^n} = [0_{\mathbb{F}}, \ldots, 0_{\mathbb{F}}]$. Then for any vector $[\alpha_1, \ldots, \alpha_n] \in \mathbb{F}^n$

$$\begin{aligned} [\alpha_1, \ldots, \alpha_n] + \mathbf{0}_{\mathbb{F}^n} &= [\alpha_1, \ldots, \alpha_n] + [0_{\mathbb{F}}, \ldots, 0_{\mathbb{F}}] \\ &: \; = [\alpha_1 + 0_{\mathbb{F}}, \ldots, \alpha_n + 0_{\mathbb{F}}] \\ &: \; = [\alpha_1, \ldots, \alpha_n] \quad \text{because } 0_{\mathbb{F}} \text{ is the additive identity in } \mathbb{F} \end{aligned}$$

(iv) Existence of Additive Inverses
We need to show that for each vector $\mathbf{v} \in \mathbb{F}^n$ there exists another vector $(-\mathbf{v}) \in \mathbb{F}^n$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}_{\mathbb{F}^n}$. Let $\mathbf{v} = [\alpha_1, \ldots, \alpha_n]$ and set $-\mathbf{v} = [-\alpha_1, \ldots, -\alpha_n]$. The latter expression makes sense since each element $\alpha_i \in \mathbb{F}$ has an additive inverse. Then

$$\begin{aligned} \mathbf{v} + (-\mathbf{v}) &= [\alpha_1, \ldots, \alpha_n] + [-\alpha_1, \ldots, -\alpha_n] \\ &: \; = [\alpha_1 + (-\alpha_1), \ldots, \alpha_n + (-\alpha_n)] \\ &= [0_{\mathbb{F}}, \ldots, 0_{\mathbb{F}}] \\ &: \; = \mathbf{0}_{\mathbb{F}^n} \end{aligned}$$

(v) Associativity and Compatibility of Scalar Multiplication
We need to show that if $\lambda, \mu \in \mathbb{F}$, then $\lambda(\mu\mathbf{v}) = (\lambda\mu)\mathbf{v}$ for all $\mathbf{v} \in \mathbb{F}^n$. Let $\mathbf{v} = [\alpha_1, \ldots, \alpha_n]$. Then

$$\begin{aligned} \lambda(\mu\mathbf{v}) &= \lambda(\mu[\alpha_1, \ldots, \alpha_n]) \\ &= \lambda([\mu\alpha_1, \ldots, \mu\alpha_n]) \\ &= [\lambda(\mu\alpha_1), \ldots, \lambda(\mu\alpha_n)] \\ &= [(\lambda\mu)\alpha_1, \ldots, (\lambda\mu)\alpha_n] \quad \text{by associativity of multiplication in } \mathbb{F} \\ &: \; = (\lambda\mu)[\alpha_1, \ldots, \alpha_n] \\ &= (\lambda\mu)\mathbf{v} \end{aligned}$$

(vi) Distibutativity of Scalar Multiplication over Addition of Scalars
We need to show that if $\lambda, \mu \in \mathbb{F}$ and $\mathbf{v} \in \mathbb{F}^n$ that $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$. Let $\mathbf{v} = [\alpha_1, \ldots, \alpha_n]$. Then

$$\begin{aligned}
(\lambda + \mu)\mathbf{v} &= (\lambda + \mu)[\alpha_1, \ldots, \alpha_n] \\
&: = [(\lambda + \mu)\alpha_1, \ldots, (\lambda + \mu)\alpha_n] \\
&= [\lambda\alpha_1 + \mu\alpha_1, \ldots, \lambda\alpha_n + \mu\alpha_n] \qquad \text{by distributive law in } \mathbb{F} \\
&: = [\lambda\alpha_1, \ldots, \lambda\alpha_n] + [\mu\alpha_1, \ldots, \mu\alpha_n] \\
&: = \lambda[\alpha_1, \ldots, \alpha_n] + \mu[\alpha_1, \ldots, \alpha_n] \\
&= \lambda\mathbf{v} + \mu\mathbf{v}
\end{aligned}$$

(vii) Distributivity of Scalar Multiplication over Vector Addition
We need to show that if $\lambda \in \mathbb{F}$ and $[\alpha_1, \ldots, \alpha_n], [\beta_1, \ldots, \beta_n] \in \mathbb{F}^n$ then $\lambda([\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n]) = \lambda[\alpha_1, \ldots, \alpha_n] + \lambda[\beta_1, \ldots, \beta_n]$

$$\begin{aligned}
\lambda([\alpha_1, \ldots, \alpha_n] + [\beta_1, \ldots, \beta_n]) &: = \lambda[\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n] \\
&: = [\lambda(\alpha_1 + \beta_1), \ldots, \lambda(\alpha_n + \beta_n)] \\
&= [\lambda\alpha_1 + \lambda\beta_1, \ldots, \lambda\alpha_n + \lambda\beta_n] \qquad \text{by distributativity of multiplication over addition in } \mathbb{F} \\
&= [\lambda\alpha_1, \ldots, \lambda\alpha_n] + [\lambda\beta_1, \ldots, \lambda\beta_n] \\
&= \lambda[\alpha_1, \ldots, \alpha_n] + \lambda[\beta_1, \ldots, \beta_n]
\end{aligned}$$

(viii) Scalar Multiplication by $1_{\mathbb{F}}$
We have for any $\mathbf{v} = [\alpha_1, \ldots, \alpha_n] \in \mathbb{F}^n$

$$1_{\mathbb{F}}[\alpha_1, \ldots, \alpha_n] := [1_{\mathbb{F}}\alpha_1, \ldots, 1_{\mathbb{F}}\alpha_n] = [\alpha_1, \ldots, \alpha_n]$$

and so scalar multiplication by the multiplicative identiy $1_{\mathbb{F}}$ in $\mathbb{F}$ acts trivially.

□

2. Let $\mathcal{C}^1(\mathbb{R})$ be the set of continuous, differentiable functions on the real line with values in $\mathbb{R}$. Define scalar multiplication and vector addition on $\mathcal{C}(\mathbb{R})$ by

$$\begin{aligned}
(\lambda \cdot f)(x) &: = \lambda f(x) \quad , \quad \forall\, \lambda \in \mathbb{R} \quad , \quad \forall\, f \in \mathcal{C}^1(\mathbb{R}); \\
(f + g)(x) &: = f(x) + g(x) \quad , \quad \forall\, f, g \in \mathcal{C}^1(\mathbb{R}) \ .
\end{aligned}$$

Check that $\mathcal{C}^1(\mathbb{R})$ satisfies the axioms for a vector space over $\mathbb{R}$.

- Below we check the axioms. In the computations below we constant use the circumstance that two functions in $\mathcal{C}^1(\mathbb{R})$ coincide if they have exactly the same value at each point $x \in \mathbb{R}$.

(i) Commutativity of Vector Addition

$$\begin{aligned}
(f + g)(x) &: = f(x) + g(x) \\
&= g(x) + f(x) \qquad \text{because addition in } \mathbb{R} \text{ is commutative} \\
&: = (g + f)(x)
\end{aligned}$$

(ii) Associativity of Vector Addition

$$\begin{aligned}
((f + g) + h)(x) &: = (f + g)(x) + h(x) \\
&: = (f(x) + g(x)) + h(x) \\
&= f(x) + (g(x) + h(x)) \qquad \text{because addition in } \mathbb{R} \text{ is associative} \\
&: = f(x) + (g + h)(x) \\
&: = (f + (g + h))(x)
\end{aligned}$$

(iii) Existence of Additivity Identity.

Set $\mathbf{0}_{\mathcal{C}^1(\mathbb{R})}$ to be the function on $\mathbb{R}$ with constant value 0. Then for any function $f \in \mathcal{C}^1(\mathbb{R})$

$$\begin{aligned} \left(f + \mathbf{0}_{\mathcal{C}^1(\mathbb{R})}\right)(x) \quad &: = f(x) + \mathbf{0}_{\mathcal{C}^1(\mathbb{R})}(x) \\ &= f(x) + 0 \\ &= f(x) \end{aligned}$$

(iv) Existence of Additive Inverses
We need to show that for each function $f \in \mathcal{C}^1(\mathbb{R})$ there exists another function $-f \in \mathcal{C}^1(\mathbb{R})$ such that $f + (-f) = \mathbf{0}_{\mathcal{C}^1(\mathbb{R})}$. Define the function $-f$ by $f(x) = (-1) * f(x)$. Then

$$(f + (-f))(x) = f(x) + (-f)(x) = f(x) + (-1) * f(x) = 0$$

Since $f + (-f)$ vanishes for all x, it must coincide with the additive identity $\mathbf{0}_{\mathcal{C}^1(\mathbb{F})}$ defined in (iii).

(v) Associativity and Compatibility of Scalar Multiplication
We need to show that if $\lambda, \mu \in \mathbb{R}$ then $\lambda(\mu f) = (\lambda\mu) f$ for all $f \in \mathcal{C}^1(\mathbb{R})$. We have

$$\begin{aligned} (\lambda(\mu f))(x) &= \lambda \cdot ((\mu f)(x)) \\ &= \lambda \cdot (\mu \cdot f(x)) \\ &= (\lambda\mu) \cdot f(x) \\ &= ((\lambda\mu) f)(x) \end{aligned}$$

(vi) Distibutativity of Scalar Multiplication over Addition of Scalars
We need to show that if $\lambda, \mu \in \mathbb{R}$ and $f \in \mathcal{C}^1(\mathbb{R}\ )$ then $(\lambda + \mu) f = \lambda f + \mu g$. We have

$$\begin{aligned} ((\lambda + \mu) f)(x) \quad &: = (\lambda + \mu) \cdot f(x) \\ &= \lambda f(x) + \mu f(x) \qquad \text{by distributive law in } \mathbb{R} \\ &: = (\lambda f)(x) + (\mu f)(x) \end{aligned}$$

(vii) Distributivity of Scalar Multiplication over Vector Addition
We need to show that if $\lambda \in \mathbb{R}$ and $f, g \in \mathcal{C}^1(\mathbb{R})$, then $\lambda(f + g) = \lambda f + \lambda g$. We have

$$\begin{aligned} (\lambda(f + g))(x) \quad &: = \lambda \cdot (f + g)(x) \\ &: = \lambda \cdot (f(x) + g(x)) \\ &: = \lambda f(x) + \lambda g(x) \qquad \text{by distributive law in } \mathbb{R} \\ &: = (\lambda f)(x) + (\lambda g)(x) \end{aligned}$$

(viii) Scalar Multiplication by $1 = 1_{\mathbb{R}}$
We have for any $f \in \mathcal{C}^1(\mathbb{R})$ we have

$$(1 \cdot f)(x) := 1 \cdot f(x) = f(x)$$

and so scalar multiplication by the multiplicative identiy 1 in $\mathbb{R}$ acts trivially.

□

3. Determine which of the following subsets are subspaces of $\mathcal{C}^1(\mathbb{R})$

(a) The set of polynomial functions in $\mathcal{C}^1(\mathbb{R})$.

- The set of polynomial functions on $\mathbb{R}$ form a subset of $\mathcal{C}^1(\mathbb{R})$, to show that it is in fact a subspace just need to show that the polynomial functions are closed under the operations of taking linear combinations. Suppose $f(x) = a_n x^n + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + \cdots + b_1 x + b_0$. Without loss of generality, we can assume $m = n$ (For example, if $m < n$ we can add $0 \cdot x^n + 0 \cdot x^{n-1} + \cdots + 0 \cdot x^{m+1}$ to g without changing its values as a function.) Then

$$\begin{aligned} (\alpha f + \beta g)(x) &= (a_n x^n + \cdots \alpha_1 x + a_0) + (b_n x^n + \cdots + b_1 x + b_0) \\ &= (\alpha_n + b_n) x^n + \cdots + (a_1 + b_1) x + (a_0 + b_0) \end{aligned}$$

Since the expression on the far right is a polynomial function, we conclude that the subset of polynomial functions is closed under scalar multiplication and vector addition; hence, it is a subspace of $\mathcal{C}^1(\mathbb{R})$.

(b) The set of all functions $f \in \mathcal{C}^1(\mathbb{R})$ such that $f\left(\frac{1}{2}\right)$ is a rational number.

- This is not a subspace. To be a subspace it would have to be closed under scalar multiplication by any real number. If we take any function f such that $f\left(\frac{1}{2}\right) = 1$, then f lies in the stated subset. But $\sqrt{2} \cdot f$ is not in this subset since its value at $x = \frac{1}{2}$ is $\sqrt{2} \cdot f\left(\frac{1}{2}\right) = \sqrt{2} \cdot 1 = \sqrt{2} \notin \mathbb{Q}$.

(c) The set of all $f \in \mathcal{C}^1(\mathbb{R})$ such that $f\left(\frac{1}{2}\right) = 0$.

- This is a subspace of $\mathcal{C}^1(\mathbb{R})$. To see this consider an arbitary linear combination $\alpha f + \beta g$ of such functions

$$(\alpha f + \beta(g))(x) = \alpha f(x) + \beta g(x) = \alpha \cdot 0 + \beta \cdot 0 = 0$$

Since a linear combination is always in the stated subset of $\mathcal{C}^1(\mathbb{R})$, the stated subset is a subspace of $\mathcal{C}^1(\mathbb{R})$.

(d) The set of all $f \in \mathcal{C}^1(\mathbb{R})$ such that $\int_0^1 f(x)\,dx = 1$

- This is not a subspace, since it is closed neither under scalar multiplication or vector addition. Explicitly, if f, g belong to this subset of $\mathcal{C}^1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ satisfies

$$\int_0^1 (\alpha f + \beta g)(x)\,dx = \alpha \int_0^1 f(x)\,dx + \beta \int_0^1 g(x)\,dx = \alpha \cdot 1 + \beta \cdot 1 \neq 1 \quad \text{in general}$$

(e) The set of all $f \in \mathcal{C}^1(\mathbb{R})$ such that $\int_0^1 f(x)\,dx = 0$

- This a subspace. Let f, g belong to this subset of $\mathcal{C}^1(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ satisfies

$$\int_0^1 (\alpha f + \beta g)(x)\,dx = \alpha \int_0^1 f(x)\,dx + \beta \int_0^1 g(x)\,dx = \alpha \cdot 0 + \beta \cdot 0 \neq 0 \quad \text{always}$$

and so the subset is closed under scalar multiplication and vector addition; hence it is a subspace.

(f) The set of all $f \in \mathcal{C}^1(\mathbb{R})$ such that $\frac{df}{dt} = 0$.

- This is a subspace of $\mathcal{C}^1(\mathbb{R})$. Suppose f, g belong to this subset and $\alpha, \beta \in \mathbb{R}$. Then

$$\left(\frac{d}{dx}(\alpha f + \beta g)\right) = \alpha \frac{df}{dx} + \beta \frac{dg}{dx} = \alpha \cdot 0 + \beta \cdot 0 = 0$$

and the so the stated subset is closed under scalar multiplication and vector addition; hence it is a subspace.

4. Prove that a subspace (a subset of a vector space that is closed under scalar mulitiplication and vector addition) is itself a vector space by verifying all 8 axioms.

- Here it is to be understood that the operations of scalar mulitiplication and vector addition in S are just the restrictions of the corresponding operations in V. Thus,

$$\begin{aligned} +_S &: S \times S \to S &=& \; +_V|_S \\ *_S &: \mathbb{F} \times S \to S &=& \; *_V|_S \end{aligned}$$

The images of these restrictions is indeed S since S is closed under scalar multiplication and vector addition. (Here I'm using $+_V$and $*_V$ to indicate addition and scalar multiplication of vectors in V, and distinguishing that, at least notationally, from the addition and scalar multiplication of vectors in S.)

Verification of the Axioms:

(i) $v +_S u = u +_S v$

$$\begin{aligned} v +_S u &\equiv v +_V u && \text{defn. of addition in } S \\ &= u +_V v && \text{since } V \text{ is a vector space} \\ &= u +_S v && \text{defn. of addition in } S \end{aligned}$$

(ii) $(v +_S u) +_S w = v +_S (u +_S w)$

$$\begin{aligned} (v +_S u) +_S w &= (v +_V u) +_V w && \text{defn. of addition in } S \\ &= v +_V (u +_V w) && \text{since } V \text{ is a vector space} \\ &= v +_S (u +_S w) && \text{defn. of addition in } S \end{aligned}$$

(iii) Existence of Additivity Identity.
Since S is closed under scalar multiplication, if we start with an $v \in S$, $0_{\mathbb{F}} *_S v \in S$. But

$$0_{\mathbb{F}} \cdot_S v = 0_{\mathbb{F}} *_V v = \mathbf{0}_V .$$

So $\mathbf{0}_V \in S$. Moreover, for any vector $u \in S$

$$u +_S \mathbf{0}_V = u +_V \mathbf{0}_V = u$$

and so $\mathbf{0}_V$ is also an additive identity in S.

(iv) Existence of Additive Inverses
Since S is closed under scalar multiplication, if $v \in S$, so is $(-1_{\mathbb{F}} *_S v)$. But

$$\begin{aligned} v +_S (-1_{\mathbb{F}} *_S v) &= v +_V (-1_{\mathbb{F}} *_V v) \\ &= (1_F *_V v) +_V (-1_{\mathbb{F}} * v) \\ &= (1_{\mathbb{F}} +_{\mathbb{F}} (-1_{\mathbb{F}})) *_V v \\ &= 0_{\mathbb{F}} *_V v \\ &= \mathbf{0}_V \\ &= \mathbf{0}_S \end{aligned}$$

(v) $(\lambda\mu) *_S v = \lambda (\mu *_S v)$ for all $\lambda, \mu \in \mathbb{F}$ and all $v \in S$:

$$\begin{aligned} (\lambda\mu) *_S v &= (\lambda\mu) *_V v && \text{defn of scalar mult. in } S \\ &= \lambda *_V (\mu *_V v) && \text{since } V \text{ is a vector space} \\ &= \lambda *_S (\mu *_S v) && \text{defn. of scalar mult. in } S \end{aligned}$$

(vi) $(\lambda + \mu) *_S v = (\lambda *_S v) +_S (\mu *_S v)$ for all $\lambda, \mu \in \mathbb{F}$ and all $v \in S$:

$$\begin{aligned} (\lambda + \mu) *_S v &= (\lambda + \mu) *_V v && \text{defn. of scalar mult. in } S \\ &= (\lambda *_V v) +_V (\mu *_V v) && \text{since } V \text{ is a vector space} \\ &= (\lambda *_S v) +_S (\mu *_S v) && \text{defn of scalar mult and vector addition in } S \end{aligned}$$

(vii) $\lambda *_S (v +_S w) = (\lambda *_S v) +_S (\lambda *_S w)$ for all $\lambda \in \mathbb{F}$ and all $v, w \in S$:

$$\begin{aligned} \lambda *_S (v +_S w) &= \lambda *_V (v +_V w) && \text{defn. of scalar mult. and vector addition in } S \\ &= (\lambda *_V v) +_V (\lambda *_V w) && \text{since } V \text{ is a vector space} \\ &= (\lambda *_S v) +_S (\lambda *_S w) && \text{defn. of scalar mult. and vector addition in } S \end{aligned}$$

(viii) $1_{\mathbb{F}} *_S v = v$ for all $v \in S$:

$$\begin{aligned} 1_{\mathbb{F}} *_S v &= 1_{\mathbb{F}} *_V v && \text{defn. of scalar mult. in } S \\ &= v && \text{since } V \text{ is a vector space} \end{aligned}$$

□

5. Is the intersection of two subspaces a subspace (prove your answer)?

- Yes. Let W, U be two subspaces of a vector space V. The intersection of W and U is

$$W \cap U = \{v \in V \mid v \in W \text{ and } v \in U\} .$$

Let u, v be any two vectors in $W \cap U$, and let $\alpha, \beta \in \mathbb{F}$. Consider the linear combination $\alpha v + \beta u$. Because, $u, v \in W \cap U$, in particular, both u and v live in the subspace W. Since W is a subspace, we have $\alpha u + \beta v \in W$. On the other hand, both u and v live in U, and so because U is a subspace, $\alpha u + \beta v$ lies in U. Thus, $\alpha u + \beta v$ lies in both W and U and so it lies in $W \cap U$. Thus, the intersection of two subspaces is closed under scalar multiplication and vector addition; hence it too is a subspace.

□

6. Is the union of two subspaces a subspace (explain your answer)?

- No. A single counter-example can justify this claim. Consider the x and y axes of the usual Cartesian plane $\mathbb{R}^2$.

$$\begin{aligned}\ell_x &= \{[x,0] \mid x \in \mathbb{R}\} \\ \ell_y &= \{[0,y] \mid y \in \mathbb{R}\}\end{aligned}$$

Each axis is a 1-dimensional subspace of $\mathbb{R}^2$. We have

$$\ell_x \cup \ell_y = \{\mathbf{v} = [s,0] \text{ or } [0,s] \mid \text{for some } s \in \mathbb{R}\}$$

Then both $[1,0]$ and $[0,1]$ lie in $\ell_x \cup \ell_y$. But

$$[1,0] + [0,1] = [1,1] \notin \ell_x \cup \ell_y$$

So $\ell_x \cup \ell_y$ is not a subspace.

7. Show that a set of vectors which contains a linearly dependent set of vectors is itself a linearly dependent set of vectors.

- Let $S = \{v_1, \ldots, v_k\}$ be a linear dependent set of vectors and let $T = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_m\}$ be another set of vectors containing S. Because S is a linearly dependent set, there must be a dependence relation

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = \mathbf{0}_V$$

with not all $\alpha_k = 0_{\mathbb{F}}$. But then

$$\alpha_1 v_1 + \cdots + \alpha_k v_k + 0_{\mathbb{F}} \cdot v_{k+1} + 0_{\mathbb{F}} \cdot v_{k+2} + \cdots + 0_{\mathbb{F}} \cdot v_m = \mathbf{0}_V$$

and because we know at least one of the α_i , $1 \leq i \leq k$, is not equal to $0_{\mathbb{F}}$ this provides a dependence relation for T. So the set T is a linearly dependent set as well.

8. Let $\{v_1, \ldots, v_n\}$ be a basis for a (non-trivial) vector space V. Show that $v_i \neq \mathbf{0}_V$ for all $i = 1, \ldots, n$.

- Suppose $v_i = \mathbf{0}_V$. Then

$$\begin{aligned}0_{\mathbb{F}} \cdot v_1 + 0_{\mathbb{F}} \cdot v_2 + \cdots + 0_{\mathbb{F}} \cdot v_{i-1} + 1_{\mathbb{R}} \cdot v_i + 0_{\mathbb{F}} \cdot v_{i+1} + \cdots + 0_{\mathbb{F}} \cdot v_n &= \mathbf{0}_V + \mathbf{0}_V + \cdots + \mathbf{0}_V + 1 \cdot \mathbf{0}_V + \mathbf{0}_V + \cdots + \mathbf{0}_V \\ &= \mathbf{0}_V\end{aligned}$$

Since the coefficient of v_i on the extreme left is non-zero, this identity furnishes us with a dependence relation for $\{v_1, \ldots, v_i, \ldots, v_n\}$. Hence this set of vector is linearly dependent. But the vectors in a basis must be linearly independent. Thus, $\{v_1, \ldots, v_i, \ldots, v_n\}$ cannot be a basis.

□

9. Let $\{v_1, \ldots, v_k\}$ be a linearly independent set of vectors. Let

$$\begin{aligned}u &= \alpha_1 v_1 + \cdots + \alpha_k v_k \\ w &= \beta_1 v_1 + \cdots + \beta_k v_k\end{aligned}$$

Prove that $u = w$ if and only if $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$.

- $\Rightarrow$ Suppose $u = w$. Then $u - w = \mathbf{0}_V$. But then we have

(*) $$\mathbf{0}_V = u - w = (\alpha_1 - \beta_1)\, v_1 + \cdots + (\alpha_k - \beta_k)\, v_k$$

Now if any of the coefficients $\alpha_i - \beta_i$ on the right are non-zero, then (*) will furnish us with a dependence relation for the set $\{v_1, \ldots, v_k\}$. But by hypothesis, the vectors $\{v_1, \ldots, v_k\}$ are linearly independent − and so we'll have a contradition unless each $\alpha_i - \beta_i = 0_{\mathbb{F}}$. But this just means we must take $\alpha_i = \beta_i$ for each i, $1 \leq i \leq k$.

$\Longleftarrow$ Suppose $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_k = \beta_k$. Then we have

$$\begin{aligned}u &= \alpha_1 v_1 + \cdots + \alpha_k v_k \\ &= \beta_1 v_1 + \cdots + \beta_k v_k \\ &= w\end{aligned}$$

□

10. Show that $\{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space $\mathcal{P}_n$ of polynomials of degree $\leq n$. (Hint: just check that the definition of a basis is satisfied.)

- We need to show that $\{1, x, x^2, \ldots, x^n\}$ is a linearly independent set of generators for $\mathcal{P}_n$. By definition, every polynomial $p = a_0 + a_1 x + \cdots + a_n x^n$ of degree $\leq n$ is a linear combination of $1, x, \ldots, x^n$; so
$$\mathcal{P}_n = span\,(1, x, \ldots, x^n)$$
and $\{1, x, \ldots, x^n\}$ is a set of generators for $\mathcal{P}_n$. The zero vector in $\mathcal{P}_n$ is the zero polynomial
$$\mathbf{0}_{\mathcal{P}_n} = 0 + 0 \cdot x + 0 \cdot x^2 + \cdots + 0 \cdot x^n$$
Since two polynomials are equal if and only if all their coefficients coincide
$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \mathbf{0}_{\mathcal{P}_n}$$
requires
$$a_0 = 0\ ,\ a_1 = 0\ ,\ a_2 = 0\ ,\ \ldots\ ,\ a_n = 0 \quad .$$
Thus, the set $\{1, x, \ldots, x^n\}$ is a linearly independent set of generators for $\mathcal{P}_n$ and hence a basis for $\mathcal{P}_n$.