LECTURE 2

Systems of First Order ODEs with Constant Coefficients

Okay, now with our review of linear algebra completed, we can begin to solve systems of homogeneous, first order, differential equations.

Recall that an ordinary differential equation is a differential equation in which there is only one underlying variable. An ordinary differential equation is **linear** if it can be written in the form

$$\frac{d^n y}{dt^n} + p_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_1(t)\frac{dy}{dt} + p_0(t)\,y = g(t)$$

where $p_{n-1}(t), \ldots, p_1(t), p_0(t)$ and $g(t)$ are functions of the underlying variable x. A linear first order differential equation is one of the form

$$y' + p(t)\,y = g(t) \quad .$$

Recall that the general solution of such an equation is given by

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)\,g(t)\,dt + \frac{C}{\mu(t)}$$

where

$$\mu(t) := \exp\left(\int p(t)\,dt\right)$$

The very easiest case is when the function $p(t)$ is just a constant $-\lambda$ and $g(t) = 0$. In the case, we have

$$y' = \lambda y \quad \Longrightarrow \quad y = Ce^{\lambda x} \quad .$$

There are a couple of ways to generalize this simplest example of a first order ordinary differential equations; one can consider higher order ordinary linear differental equations, or one can consider linear differential equations where there is more than one underlying variable (i.e., a first order, linear, partial differential equation). We will begin this course by considering first order ordinary differential equations in which more than one unknown function occurs.

DEFINITION 2.1. *An $n\times n$ **system of first order linear ODEs** is a set of n differential equations involving n unknown functions $x_1, \ldots, x_m$ of the form*

$$\begin{aligned}
\frac{dx_1}{dt} - a_{11}(t)\,x_1(t) - a_{12}(t)\,x_2(t) - \cdots - a_{1n}(t)\,x_n(t) &= g_1(t) \\
\frac{dy_2}{dt} - a_{21}(t)\,x_1(t) - a_{22}(t)\,x_2(t) - \cdots - a_{2n}(t)\,x_n(t) &= g_2(t) \\
&\vdots \\
\frac{dy_n}{dt} - a_{n1}(t)\,x_1(t) - a_{n2}(t)\,x_2(t) - \cdots - a_{nn}(t)\,x_n(t) &= g_n(t)
\end{aligned}$$

*We say that such a system is **homogeneous** if each of the functions $g_1(t), \ldots, g_n(t)$ is just the constant function 0. We say that such a systems has **constant coefficients** if each of the coefficient functions $a_{ij}(t)$, $1 \le i, j \le n$, is a constant function.*

Alternatively, an $n \times n$ homogeneous linear system of first order ODEs is a system of differential equations of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \quad \Longleftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

EXAMPLE 2.2. Suppose $\mathbf{A}$ is a diagonal $n \times n$ matrix

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

Then the homogeneous linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

is easily solved. For the differential equations governing the n unknown functions are completely independent of each other and easily solved

$$\begin{aligned} \frac{dx_1}{dt} &= \lambda_1 x_1 \quad \Longrightarrow \quad x_1(t) = c_1 e^{\lambda_1 t} \quad , \quad c_1 \text{ a constant} \\ &\vdots \\ \frac{dx_n}{dt} &= \lambda_n x_n \quad \Longrightarrow \quad x_n(t) = c_n e^{\lambda_n t} \quad , \quad c_n \text{ a constant} \end{aligned}$$

The general solution is thus

$$\mathbf{x}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

REMARK 2.3. When the coefficient matrix $\mathbf{A}$ is diagonal, we say that the system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$ is **decoupled**.

Let's now consider the general case of an $n \times n$ homogeneous linear system with constant coefficients.

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{1}$$

We will solve such systems by simply making a change of variables so that the differential equations completely decouple and then solve the corresponding decoupled system as in the preceding example.

Here's a sketch of how this will work. Suppose we had an invertible matrix $\mathbf{C}$ such that

$$\mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$\mathbf{D}$ being a diagonal matrix. Then as above we could simply write down the general solution of $\dot{\mathbf{z}} = \mathbf{D}\mathbf{z}$ as

$$\begin{aligned} z_1 &= C_1 e^{\lambda_1 t} \\ z_2 &= C_2 e^{\lambda_2 t} \\ &\vdots \\ z_n &= C_n e^{\lambda_n t} \end{aligned}$$

Let $\mathbf{z}(t)$ be such a solution, and consider the vector $\mathbf{x}(t)$ obtained by multiplying $\mathbf{z}(t)$ from the left by the matrix $\mathbf{C}$

$$\mathbf{x} = \mathbf{Cz}$$

Then

$$\frac{d\mathbf{x}}{dt} = \frac{d}{dt}(\mathbf{Cz}) = \mathbf{C}\frac{d}{dt}\mathbf{z} = \mathbf{C}\dot{\mathbf{z}} = \mathbf{C}(\mathbf{Dz}) = \mathbf{C}\left(\mathbf{C}^{-1}\mathbf{AC}\right)\mathbf{z} = \left(\mathbf{CC}^{-1}\right)\mathbf{ACz} = \mathbf{IACz} = \mathbf{A}(\mathbf{Cz}) = \mathbf{Ax}$$

In other words, $\mathbf{x} = \mathbf{Cz}$ will be a solution of our original differential equation.

$$\dot{\mathbf{y}} = \mathbf{Ay}$$

Thus, systems of the form (1) can be easily solved if we can find an invertible matrix $\mathbf{C}$ such that $\mathbf{C}^{-1}\mathbf{AC}$ is a diagonal matrix.

Here is the general procedure: to solve a homogeous linear system of ODEs with constant coefficients

$$\frac{d\mathbf{x}}{dt} = \mathbf{Ax}$$

(i) Compute the eigenvalues and eigenvectors of the coefficient matrix $\mathbf{A}$
(ii) Use the eigenvalues and eigenvectors of $\mathbf{A}$ to, respectively, construct the diagonal matrix $\mathbf{D}$ and the *change of basis* matrix $\mathbf{C}$ such that

$$\mathbf{D} = \mathbf{C}^{-1}\mathbf{AC} \quad \Longleftrightarrow \quad \mathbf{A} = \mathbf{CDC}^{-1}$$

(iii) Write down the general solution of the decoupled system

$$\frac{d\mathbf{z}}{dt} = \mathbf{Dz} \quad \Longrightarrow \quad \mathbf{z} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{bmatrix}$$

(iv) The solution of the original (coupled) system will be

$$\mathbf{x} = \mathbf{Cz}$$

Example 2.4. Find the general solution of the following system of differential equations

$$\begin{aligned} \frac{dx_1}{dt} &= x_1 + x_2 \\ \frac{dx_2}{dt} &= 4x_1 + x_2 \end{aligned}$$

The matrix formulation of this problem would be

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And so we'll begin by finding a matrix $\mathbf{C}$ that diagonalizes $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

FIrst we find the eigenvalues of $\mathbf{A}$

$$\begin{aligned} & 0 = \det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3) \\ & \Rightarrow \quad \lambda = -1, 3 \end{aligned}$$

Next, we find the corresponding eigenvectors

$\lambda = 3:$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\mathbf{A} - (3)\,\mathbf{I})\,\mathbf{v} = \begin{bmatrix} 1-(3) & 1 \\ 4 & 1-(3) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2v_1 + v_2 \\ 4v_1 - 2v_2 \end{bmatrix}$$

$$\Rightarrow \quad 2v_1 - v_2 = 0 \quad \Rightarrow \quad v_1 = -\frac{1}{2}v_2 \quad \Rightarrow \quad \mathbf{v} = v_2 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \approx v_2' \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(In the last step, we simply absorbed an ugly factor of 1/2 into a (redefined) free parameter $v_2' \equiv \frac{1}{2}v_2$.)

$\lambda = -1:$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = (\mathbf{A} - (-1)\,\mathbf{I})\,\mathbf{v} = \begin{bmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 + v_2 \\ 4v_1 + 2v_2 \end{bmatrix}$$

$$\Rightarrow \quad 2v_1 + v_2 = 0 \quad \Rightarrow \quad v_1 = -\frac{1}{2}v_2 \quad \Rightarrow \quad \mathbf{v} = v_2 \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \approx v_2' \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Having found the eigenvectors and eigenvalues of $\mathbf{A}$ we can now write down the matrices $\mathbf{C}$ and $\mathbf{D}$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \quad , \quad \mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

The general solution of

$$\frac{d\mathbf{z}}{dt} = \mathbf{D}\mathbf{z}$$

will be

$$\mathbf{z} = \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix}$$

And so the general solution of $\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y}$ will be

$$\begin{aligned} \mathbf{y} = \mathbf{C}\mathbf{z} &= \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{bmatrix} =: \begin{bmatrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{bmatrix} \\ &= c_1 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{aligned}$$

Analysis of Solutions: Above we expressed the general solution of

$$\begin{bmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

in terms of two linearly independent solutions

$$\begin{aligned} \mathbf{x}_1(t) &= e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \mathbf{x}_2(t) &= e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \end{aligned}$$

Notice in either case the "trajectories" $\mathbf{x}_i(t)$ of a solution are just simple half lines (they're just scalar multiples of constant vectors).

A more general trajectory will curve about in the (x_1, x_2)-plane. To get in idea of what these other solutions should look like, it is convenient to plot the **direction field** associated with the system.

Here's how that works. If

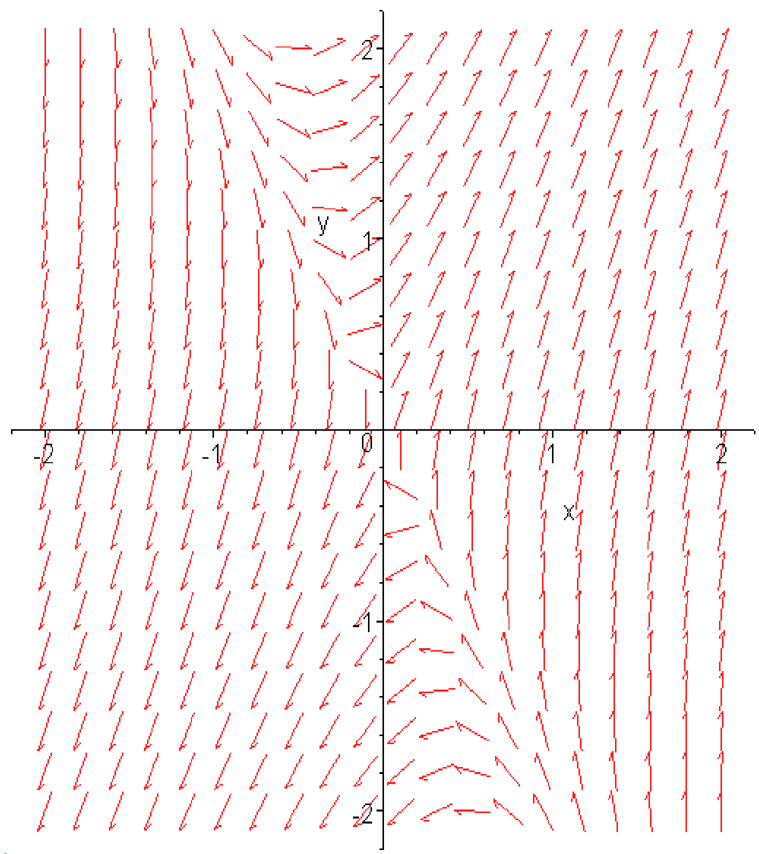
$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

is our system of ODEs, with $\mathbf{A}$ a constant matrix, then knowing that a solution passes through a given $\mathbf{x}$ means we know the tangent vector to the solution curve at that point - because all we have to do is evaluate the right hand side of (*) at $\mathbf{x}$ (that is, carry out the matrix multiplication $\mathbf{A}\mathbf{x}$).

So pick a grid of points $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ in the (x_1, x_2)-plane, construct a table

$$\begin{array}{ll} \mathbf{x}_1 & \mathbf{A}\mathbf{x}_1 \\ \mathbf{x}_2 & \mathbf{A}\mathbf{x}_2 \\ \vdots & \vdots \\ \mathbf{x}_k & \mathbf{A}\mathbf{x}_k \end{array}$$

and then plot each of the points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ and then attach to each of these points a small arrow in the direction of, respectively, $\mathbf{A}\mathbf{x}_1, \ldots, \mathbf{A}\mathbf{x}_k$. You will get something looking like this:



The figure above is actually the direction field plot for differential equation in Example 2.4.

1. Fundamental Matrix

As in the preceding example, we can always express the general solution of an $n \times n$ linear system

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$$

as a linear combination of *fundamental solutions*

$$(*) \qquad \mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots c_n\mathbf{x}^{(2)}(t)$$

where the vector-valued functions $\mathbf{x}^{(i)}(t)$ are linearly independent vector-valued functions of t. Here *linearly independent* is just the usual notion

$$\mathbf{0} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots c_n\mathbf{x}^{(2)}(t) \quad \Longrightarrow \quad c_1 = 0,\, c_2 = 0,\, \ldots\, ,\, c_n = 0$$

which amounts the condition that

$$0 \neq \det \begin{bmatrix} x_1^{(1)}(t) & x_1^{(2)}(t) & \cdots & x_1^{(n)}(t) \\ x_2^{(1)}(t) & x_2^{(2)}(t) & \cdots & x_2^{(n)}(t) \\ \vdots & \vdots & \ddots & \ddots \\ x_n^{(1)}(t) & x_n^{(2)}(t) & \cdots & x_n^{(n)}(t) \end{bmatrix} \qquad \forall\, t$$

Another way of *presenting* the general solution is in terms of a *fundamental matrix* $\mathbf{\Phi}$. This done by arranging the n linearly independent vector solutions as the colums of a matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} | & | & & | \\ \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) & \cdots & \mathbf{x}^{(n)}(t) \\ | & | & & | \end{bmatrix}$$

and then representing the general solution as the matrix product of $\mathbf{\Phi}(t)$ with an arbitary n-dimensional constant column vectors

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\,\mathbf{c} = \begin{bmatrix} | & | & & | \\ \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) & \cdots & \mathbf{x}^{(n)}(t) \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Of course it is a familar identity of matrix multiplication that

$$\begin{bmatrix} | & | & & | \\ \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) & \cdots & \mathbf{x}^{(n)}(t) \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots c_n\mathbf{x}^{(2)}(t)$$

so writing $\mathbf{x}(t) = \mathbf{\Phi}(t)\,\mathbf{c}$ is essentially the same thing as (*). However, it is sometimes to useful to think of $\mathbf{\Phi}(t)$ as a time dependent matrix which transports a constant vector $\mathbf{c}$ corresponding to certain initial vector to its position at time t.

2. Complex Eigenvalues

Consider now the system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \mathbf{x}$$

We have

$$\det \begin{bmatrix} -\frac{1}{2} - \lambda & 1 \\ -1 & -\frac{1}{2} - \lambda \end{bmatrix} = \lambda^2 + \lambda + \frac{5}{4}$$

and so the roots of the characteristic polynomial are

$$\lambda = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(5/4)}}{2(1)} = \frac{-1 \pm \sqrt{-4}}{2} = -\frac{1}{2} + i\ ,\ -\frac{1}{2} - i$$

The corresponding eigenvectors are:

$\lambda = -\frac{1}{2} + i$

$$\begin{bmatrix} -\frac{1}{2} - \left(-\frac{1}{2} + i\right) & 1 \\ -1 & -\frac{1}{2} - \left(-\frac{1}{2} + i\right) \end{bmatrix} = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \quad \mathbf{v}_{\lambda = -\frac{1}{2} + i} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} - \left(-\frac{1}{2} - i\right) & 1 \\ -1 & -\frac{1}{2} - \left(-\frac{1}{2} - i\right) \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$
$$\Longrightarrow \quad \mathbf{v}_{\lambda = -\frac{1}{2}+i} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

So we'll have

$$\mathbf{x} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_1 e^{\left(-\frac{1}{2}+i\right)t} \\ c_2 e^{\left(-\frac{1}{2}-i\right)t} \end{bmatrix} = c_1 e^{\left(-\frac{1}{2}+i\right)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{\left(-\frac{1}{2}-i\right)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

This is a correct formula for the general solution; however, the answer is given in terms of a pair of complex vectors muliplied by a pair of complex exponential functions. Often what one really wants is a pair of independent real vector solutions. Here's how to get such a pair.

We have, by the Euler formula,

$$e^{\left(-\frac{1}{2}\pm i\right)t} = e^{-\frac{1}{2}t} \left(\cos(t) \pm \sin(t)\right)$$

and so

$$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} c_1 e^{\left(-\frac{1}{2}+i\right)t} \\ c_2 e^{\left(-\frac{1}{2}-i\right)t} \end{bmatrix} = \begin{bmatrix} c_1 e^{\left(-\frac{1}{2}+i\right)t} + c_2 e^{\left(-\frac{1}{2}-i\right)t} \\ i c_1 e^{\left(-\frac{1}{2}+i\right)t} - i c_2 e^{\left(-\frac{1}{2}-i\right)t} \end{bmatrix}$$
$$= e^{-\frac{1}{2}t} \begin{bmatrix} (c_1 + c_2)\cos(t) + i\,(c_1 - c_2)\sin(t) \\ i\,(c_1 - c_2)\cos(t) - (c_1 + c_2)\sin(t) \end{bmatrix}$$

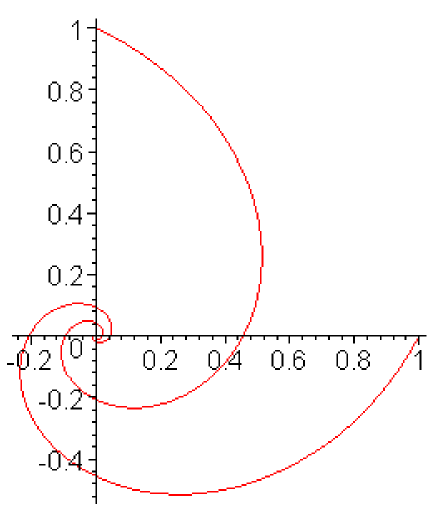
So if we take $c_1 = c_2 = \frac{1}{2}$ we get one totally real solution

$$\mathbf{x}^{(1)}(t) = e^{-\frac{1}{t}t} \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}$$

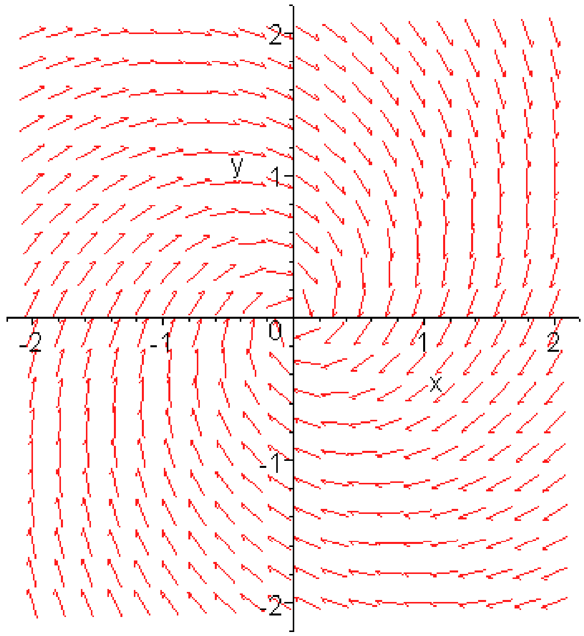
and if we take $c_1 = -c_2 = \frac{-i}{2}$, we get a separate totally real solution

$$\mathbf{x}^{(2)}(t) = e^{-\frac{1}{2}t} \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$

The trajectories of both these solutions are spirials



By the way, the direction field plot for the original system is



Let me now describe a bit more generally, how we handle the situation where the entries of **A** are real but the eigenvalues are complex $\lambda = \alpha \pm i\beta$. Now what happens in this situation is that not only are the two eigenvalues complex conjugates of one another, but also the corresponding eigenvectors (can be normalized so that they) are complex conjugates of each other. Thus, the general solution can be written (in a 2×2 case) as

$$\mathbf{x}(t) = \begin{pmatrix} e^{(\alpha+i\beta)t}\xi_1 & e^{(\alpha-i\beta)t}\overline{\xi_1} \\ e^{(\alpha+i\beta)t}\xi_2 & e^{(\alpha-i\beta)t}\overline{\xi_2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \xi_1 c_1 e^{(\alpha+i\beta)t} + \overline{\xi_1} c_2 e^{(\alpha-i\beta)t} \\ \xi_2 c_1 e^{(\alpha+i\beta)t} + \overline{\xi_2} c_2 e^{(\alpha-i\beta)t} \end{pmatrix} = e^{\alpha t} \begin{pmatrix} \xi_1 c_1 e^{i\beta t} + \overline{\xi_1} c_2 e^{-i\beta t} \\ \xi_2 c_1 e^{i\beta t} + \overline{\xi_2} c_2 e^{-i\beta t} \end{pmatrix}$$

Let if we choose $c_1 = c_2 = \frac{1}{2}$ we have, as one solution

$$\mathbf{x}_i(t) = e^{\alpha t} \begin{pmatrix} \frac{1}{2}\left(\xi_1 e^{i\beta t} + \overline{\xi_1 e^{i\beta t}}\right) \\ \frac{1}{2}\left(\xi_2 e^{i\beta t} + \overline{\xi_2 e^{i\beta t}}\right) \end{pmatrix} = \begin{pmatrix} e^{\alpha t}\operatorname{Re}\left(\xi_1 e^{i\beta t}\right) \\ e^{\alpha t}\operatorname{Re}\left(\xi_2 e^{i\beta t}\right) \end{pmatrix} = \operatorname{Re}\left(e^{(\alpha+i\beta t)}\xi\right)$$

and if we choose $c_1 = -c_2 = \frac{i}{2}$, we similarly obtain a second independent solution

$$\mathbf{x}_2(t) = \begin{pmatrix} e^{\alpha t}\operatorname{Im}\left(\xi_1 e^{i\beta t}\right) \\ e^{\alpha t}\operatorname{Im}\left(\xi_2 e^{i\beta t}\right) \end{pmatrix} = \operatorname{Im}\left(e^{(\alpha+i\beta t)}\xi\right)$$

The general solution can then be expresses as a linear combination of these two purely real-valued solutions.

3. Non-diagonalizable Systems

Consider now the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In this case, we have

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 + 1 = (\lambda - 2)^2$$

We thus have a single eigenvalue

$$\lambda = 2$$

The corresponding eigenspace is

$$NullSp\left(\begin{bmatrix} 1-2 & -1 \\ 1 & 3-2 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\right) = NullSp\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$$
$$= span\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

So we have only one eigenvector.

$$\mathbf{v}_{\lambda=2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The corresponding fundamental solution will be

$$\mathbf{x}(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{2}$$

There are other solutions though. As an ansatz for a second solution consider

$$\mathbf{x}(t) = te^{2t}\xi + e^{2t}\eta$$

where

$$\xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and η is some vector to be determined. Plugging this $\mathbf{x}(t)$ into the differential equation yields

$$e^{2t}\xi + 2te^{2t}\xi + 2e^{2t}\eta = \frac{d\mathbf{x}}{dt} = \mathbf{A}\left(te^{2t}\xi + e^{2t}\eta\right) = 2te^{2t}\xi + e^{2t}\mathbf{A}\eta$$

where on the right we used $\mathbf{A}\xi = 2\xi$. Cancelling the $2te^{2t}\xi$ terms on both sides and then dividing by e^{2t} yields

$$\xi + 2\eta = \mathbf{A}\eta$$

or

$$(\mathbf{A} - 2\mathbf{I})\,\eta = \xi \tag{3}$$

On the other hand, working backwards from (3) we see that if η satisfies

$$(\mathbf{A} - 2\mathbf{I})\,\eta = \xi$$

then

$$\mathbf{x}(t) = te^{2t}\xi + e^{2t}\eta \tag{4}$$

will be a solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t)$$

The solution of (3) can be easily found using row reduction:

$$[\mathbf{A} - 2\mathbf{I} \mid \xi] = \left[\begin{array}{cc|c} 1-2 & -1 & 1 \\ 1 & 3-2 & -1 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$
$$\Longrightarrow \begin{cases} \eta_1 + \eta_2 = -1 \\ 0 = 0 \end{cases}$$
$$\Longrightarrow \eta = \begin{bmatrix} \eta_1 \\ -1-\eta_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \eta_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \eta_1\xi$$

where η_1 is arbitrary.

If we now plug this η into (4) we get

$$\mathbf{x}^{(2)}(t) = te^{2t}\xi + e^{2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \eta_1\xi\right)$$

Notice that the last term

$$\eta_1 e^{2t}\xi$$

is just a constant multiple of our original solution (2)

$$\mathbf{x}^{(1)}(t) = e^{2t}\xi$$

So we can drop the $\eta_1 e^{2t}\xi$ when we write the general solution

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t)$$

as a linear combination of $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$. We thus arrive at

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) \quad \Longrightarrow \quad \mathbf{x}(t) = (c_1 + tc_2)\, e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Let me know both summarize and generalize the method of solving of systems of the form

(5) $$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) \qquad , \qquad \text{with } \mathbf{A} \text{ non-diagonalizable.}$$

First let me summarize the case of a simple 2×2 system

(*) $$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)$$

where $\mathbf{A}$ is a constant matrix with a single eigenvalue λ and only one linearly independent eigenvector ξ. One independent solution will be

$$\mathbf{x}_1(t) = e^{\lambda t}\xi$$

and second linear indepedent solution can be constructed as follows. Let η be the solution of

$$\mathbf{A}\eta = \xi$$

then

$$\mathbf{x}_2(t) = te^{\lambda t}\xi + e^{\lambda t}\eta$$

will be a second independent solution and the general solution of (*) will be

$$\mathbf{x}(t) = c_1 e^{\lambda t}\xi + c_2\left(te^{\lambda t}\xi + e^{\lambda t}\eta\right)$$

More generally, for an $n \times n$ system, the procedure goes as follows.

- Find the eigenvectors and eigenvalues of $\mathbf{A}$.
- Because $\mathbf{A}$ is assumed to be non-diagonalizable, there is going to be an eigenvalue r of $\mathbf{A}$ that occurs with multiplicity k (meaning the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$ has a factor of the form $(\lambda - r)^k$), but for which there are not k linearly independent eigenvectors.
- There will, however, be one eigenvector of $\mathbf{A}$ with eigenvalue λ. Call it $\xi^{(1)}$. It will be the solution of $(\mathbf{A} - r\mathbf{I})\,\xi = \mathbf{0}$.
- Once you find $\xi^{(1)}$ successively solve

  $$(\mathbf{A} - r\mathbf{I})\,\xi^{(i)} = \xi^{(i-1)}$$

  This will furnish you with a set of *generalized eigenvectors* corresponding the eigenvalue r.
- The solutions of (5) corresponding the eigenvalue $\lambda = r$ will be of the form

  $$\mathbf{x}^{(k)}(t) = \frac{1}{k!}t^k e^{rt}\xi^{(1)} + \frac{1}{(k-1)!}t^{k-1}e^{rt}\xi^{(2)} + \cdots + \xi^{(k)} \qquad , \qquad i = 1, 2, \ldots, k$$

4. Stability of Solutions

We'll now look of the behavior of solutions of 2×2 systems of the form

(6) $$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

as $t \to \pm\infty$. We'll do this via a case-by-case analysis of the various possibilities corresponding to the eigenvalues and eigenvectors of $\mathbf{A}$:

4.1. Case 1: A has two distinct real eigenvalues r_1, r_2. In such cases the general solution of (6) will be of the form

$$\mathbf{x}(t) = c_1 e^{r_1 t}\xi + c_2 e^{r_2 t}\eta$$

with ξ and η two linearly independent vectors.

4.1.1. $r_1 > \mathbf{r}_2 > 0$. In this case the first part of the solution will dominate as $t \to +\infty$, while the second term will dominate as $t \to -\infty$. But note that as $t \to -\infty$, $\mathbf{x}(t) \to 0$ because both terms are being multiplied by exponential functions that rapidly decay as $t \to -\infty$. So the solutions will asymptotically approach the line $\mathbb{R}\xi$ as $t \to +\infty$ and asymptotically approach the line $\mathbb{R}\eta$ as $t \to -\infty$.

4.1.2. $r_1 > 0 > \mathbf{r}_2$. In this case the first part of the solution will dominate as $t \to +\infty$, while the second term will dominate as $t \to -\infty$. But note that as $t \to -\infty$, $\mathbf{x}(t)$ will not be zero, rather as $t \to -\infty$, $\mathbf{x}(t)$ will head off to infinity (that is, to say $\|\mathbf{x}(t)\| \to \infty$) in the direction of η.

4.2. Case 2: A has two complex eigenvalues $r = \lambda \pm i\mu$. This case is typified by systems where (6) takes the form

$$\frac{d\mathbf{x}}{dt} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}\mathbf{x} \iff \begin{cases} \frac{dx_1}{dt} = \lambda x_1 + \mu x_2 \\ \frac{dx_2}{dt} = -\mu x_1 + \lambda x_2 \end{cases} \tag{7}$$

If we replace the components $x_1(t), x_2(t)$ of $\mathbf{x}(t)$ with their expressions in terms of polar coordinates

$$\begin{aligned} x_1(t) &= r(t)\cos(\theta(t)) \\ x_2(t) &= r(t)\sin(\theta(t)) \end{aligned}$$

we then have

$$\frac{dr}{dt}\cos(\theta) - r\sin(\theta)\frac{d\theta}{dt} = \frac{dx_1}{dt} = \lambda x_1 + \mu x_2 = \lambda r\cos(\theta) + \mu r\sin(\theta) \tag{8a}$$

$$\frac{dr}{dt}\sin(\theta) + r\cos(\theta)\frac{d\theta}{dt} = \frac{dx_2}{dt} = -\mu x_1 + \lambda x_2 = -\mu r\cos(\theta) + \lambda r\sin(\theta) \tag{8b}$$

Multiplying equation (8a), equation by $\cos(\theta)$, (8b) by $\sin(\theta)$ and then adding the results we get

$$\left(\cos^2\theta + \sin^2\theta\right)\frac{dr}{dt} = \lambda r\left(\cos^2\theta + \sin^2\theta\right)$$

or

$$\frac{dr}{dt} = \lambda r \quad \Longrightarrow \quad r = R_0 e^{\lambda t}$$

On the other hand, multiplying equation (8a) by $-\sin(\theta)$, (8b) by $\cos(\theta)$ and then adding results yields

$$r\left(\sin^2\theta + \cos^2\theta\right)\frac{d\theta}{dt} = -\mu r\left(\sin^2\theta + \cos^2\theta\right)$$

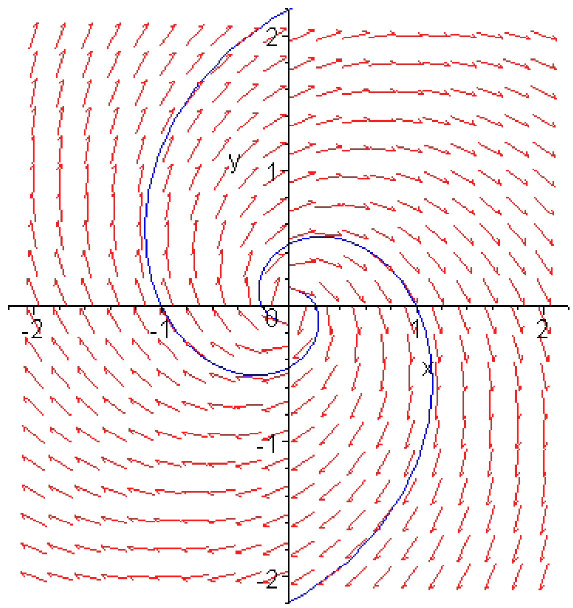
or

$$\frac{d\theta}{dt} = -\mu \quad \Longrightarrow \quad \theta(t) = -\mu t + \theta_0$$

Thus the general solution of (7) can be written

$$\begin{aligned} x_1(t) &= R_0 e^{\lambda t}\cos(-\mu t + \theta_0) \\ x_2(t) &= R_0 e^{\lambda t}\sin(-\mu t + \theta_0) \end{aligned}$$

The trajectories of such solution will be spirals that decay or expand away from the origin $\mathbf{0}$.



4.3. Case 3: A has a single eigenvalue r and a single eigenvector ζ. This case is typified by the a matrix of the form

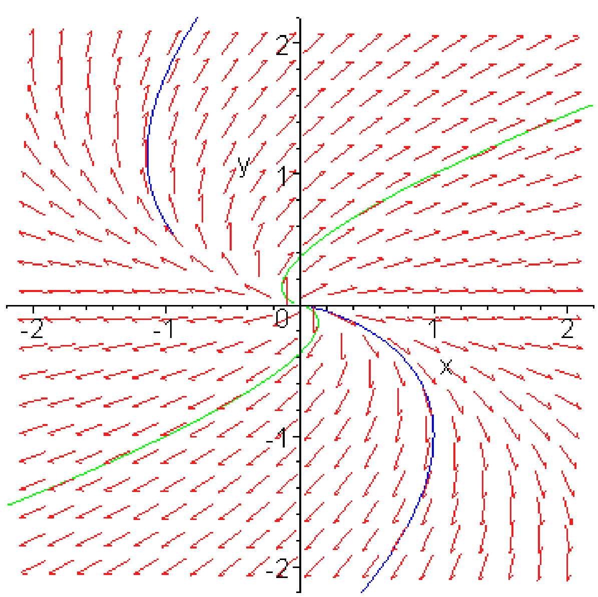
$$\mathbf{A} = \begin{pmatrix} r & \mu \\ 0 & r \end{pmatrix}$$

The general solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

will be

$$\mathbf{x}(t) = c_1 e^{rt} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(t e^{rt} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{rt} \begin{bmatrix} 0 \\ 1/\mu \end{bmatrix} \right)$$



5. Non-Autonomous Homogeneous Linear Systems

We have been considering systems of first order ordinary differential equations that can be cast in the form

$$(*) \qquad \frac{d}{dt}\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)$$

In the examples treated thus far the matrix $\mathbf{A}$ has been a constant matrix (independent of t). In such case, when the underlying variable does not appear explictly in the system of differential equations, the system is said to be *autonomous*. The system *() is special in another way: there are no terms in a system of linear differential equation that are independent of $\mathbf{x}(t)$ and its derivatives: such a system is said to be *homogeneous*. In the next lecture we shall consider non-autonomous, non-homogeneous linear systems; that is, to say systems of the general form

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{b}(t)$$

However, we have already in hand the necessary techniques to solve a non-autonomous, homogeneous system

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(t) \quad .$$

Example. Consider the linear system

$$\begin{aligned} \frac{dx_1}{dt} &= (t+1)\,x_1 + (t-1)\,x_2 \\ \frac{dx_2}{dt} &= (t-1)\,x_1 + (t+1)\,x_2 \end{aligned}$$

The relevant matrix is

$$\mathbf{A} = \begin{bmatrix} t+1 & t-1 \\ t-1 & t+1 \end{bmatrix}$$

We determine its eigenvalues as before:

$$\begin{aligned} 0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{bmatrix} t+1-\lambda & t-1 \\ t-1 & t+1-\lambda \end{bmatrix} &= \lambda^2 - 2(t+1)\lambda + 4t \\ \Rightarrow \quad \lambda &= \frac{2(t+1) \pm \sqrt{\left(4(t+1)^2 - 8t\right)}}{2} \\ \Rightarrow \quad \lambda &= (t+1) \pm \sqrt{(t+1)^2 - 4t} \\ \Rightarrow \quad \lambda &= (t+1) \pm \sqrt{t^2 - 2t + 1} \\ \Rightarrow \quad \lambda &= (t+1) \pm \sqrt{(t-1)^2} \\ \Rightarrow \quad \lambda &= 2, 2t \end{aligned}$$

So we have two eigenvalues, $\lambda - 2$ and $2t$. We can also calculate the corresponding eigenvectors they turn out to be

$$\mathbf{v}_{\lambda=2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v}_{\lambda=2t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, the matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

will convert the original coupled (non-diagonal) system to a de-coupled (diagonalize) one: as before we have

$$\mathbf{D} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C} = \begin{bmatrix} 2 & 0 \\ 0 & 2t \end{bmatrix}$$

and so if we define

$$\mathbf{y}(t) = \mathbf{C}^{-1}\mathbf{x}(t)$$

then

$$\begin{aligned} \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} \quad &\Rightarrow \quad \mathbf{C}^{-1}\frac{d}{dt}\mathbf{x} = \mathbf{C}^{-1}\mathbf{A}\mathbf{x} \\ &\Rightarrow \quad \frac{d}{dt}\left(\mathbf{C}^{-1}\mathbf{x}\right) = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}\mathbf{C}^{-1}\mathbf{x} = \left(\mathbf{C}^{-1}\mathbf{A}\mathbf{C}\right)\left(\mathbf{C}^{-1}\mathbf{x}\right) \\ &\Rightarrow \quad \frac{d}{dt}\mathbf{y} = \mathbf{D}\mathbf{y} \end{aligned}$$

The new variables y_1 and y_2 now are solutions of the decoupled system

$$\begin{bmatrix} \frac{dy_1}{dy} \\ \frac{dy_2}{dt} \end{bmatrix} = \frac{d}{dt}\mathbf{y} = \mathbf{D}(t)\,\mathbf{y} = \begin{bmatrix} 2 & 0 \\ 0 & 2t \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 \\ 2ty_2 \end{bmatrix}$$

or

$$\begin{aligned} \frac{dy_1}{dt} &= 2y_1 \\ \frac{dy_2}{dt} &= 2ty_2 \end{aligned}$$

The next step is the only difference between this non-autonomous example and the previous autonomous examples. To solve the second differential equation we have to resort to the formula

$$y' = p(x)\,y \quad \Rightarrow \quad y(x) = c\exp\left(\int p(x)\right)dx$$

(which also applies to equation for y_1). We find

$$\begin{aligned} y_1(t) &= c_1 e^{2t} \\ y_2(t) &= c_2 e^{t^2} \end{aligned}$$

We thus find the solution $\mathbf{x}(t)$ to the original system to be

$$\mathbf{x}(t) = \mathbf{C}\mathbf{y}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{2t} \\ c_2 e^{t^2} \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{t^2} \\ -c_1 e^{2t} + c_2 e^{t^2} \end{bmatrix}$$

Just as in previous examples we can reformulate this result in terms of a fundamental matrix. If we set

$$\mathbf{\Phi}(t) = \begin{bmatrix} | & | \\ y_1(t)\,\mathbf{v}_{\lambda_1} & y_2(t)\,\mathbf{v}_{\lambda_2} \\ | & | \end{bmatrix}$$

(i.e. the 2×2 matrix whose first column is the first solution $y_1(t)$ corresponding to the first eigenvalue times the first eigenvector and whose second column is the solution corresponding to the second eigenvalue times the second eigenvector), then

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad .$$