LECTURE 4

Numerical Methods for ODEs, I

I shall now give an easy method of constructing an (approximate) numerical solution to a differential equation of the form

$$\frac{dx}{dt} = F(t,x) \ , \ \forall\, t \in [a,b] \tag{4.1}$$

The beauty of this method is that it works for any first order differential equation (well, so long as the function $F(x,t)$ on the right hand side is a continuous function of x and t on the interval $[a,b]$). However, it has a rather ugly side as well - the final result will not be a presentation of the solution in terms of known functions; rather it will simply be a table of values of the solution at a discrete set of points $t_i \in [a,b]$.

To construct our numerical solution, we begin by first dividing up the interval $[a,b]$ into n subintervals. Set

$$\Delta t = \frac{b-a}{n} \tag{4.2}$$

and let

$$\begin{aligned} t_0 &= a \\ t_1 &= a + \Delta x \\ t_2 &= a + 2\Delta x \\ &\vdots \\ t_i &= a + i\Delta x \\ &\vdots \\ t_n &= a + n\Delta t = a + \frac{b-a}{\Delta t}\Delta t = b \end{aligned} \tag{4.3}$$

Let $x_i = x(t_i)$ denote the value of a solution of (4.1) at the point t_i and let $\dot{x}_i = \frac{dx}{dt}(t_i)$. The differential equation (4.1) then requires

$$\dot{x}_i = F(t_i, x_i) \ , \ i = 0, 1, \ldots, n \tag{4.4}$$

Now by making Δt small enough, we can approximate $\dot{x}_i = \frac{dx}{dt}(t_i)$ to an arbitrarily high degree of accuracy by setting

$$\dot{x}_i = \frac{dx}{dt}(t_i) \approx \frac{\Delta x}{\Delta t} = \frac{x_{i+1} - x_i}{\Delta t} \tag{4.5}$$

And so, the differential equation (4.1) is approximately equivalent to the following set of algebraic equations

$$\frac{x_{i+1} - x_i}{\Delta t} = F(t_i, x_i) \ , \ i = 0, \ldots, n-1 \tag{4.6}$$

Solving (4.6) for x_{i+1}, we obtain

$$x_{i+1} = x_i + \Delta t F(t_i, x_i) \ . \ i = 0, 1, \ldots, n-1 \tag{4.7}$$

or, more explictly,

$$x_1 = x_0 + \Delta t F(t_0, x_0) \tag{4.8}$$
$$x_2 = x_1 + \Delta t F(t_1, x_1) \tag{4.9}$$
$$x_3 = x_2 + \Delta t F(t_2, x_2) \tag{4.10}$$
$$\vdots \tag{4.11}$$
$$x_{i+1} = x_i + \Delta t F(t_i, x_i) \tag{4.12}$$
$$\vdots \tag{4.13}$$
$$x_n = x_{n-1} + \Delta t F(t_{n-1}, x_{n-1}) \tag{4.14}$$

This set of equations relates now relates all the x_i, $i = 1, 2, \ldots, n$ to x_0.

To see this, note that when $i = 0$ equation (4.8) implies

$$x_1 = x_0 + F(t_0, x_0) \tag{4.15}$$

But now inserting this expression for x_1 into the right hand side of (4.9) yields

$$x_2 = x_0 + F(t_0, x_0) + F(t_1, x_0 + F(t_0, x_0)) \tag{4.16}$$

Thus, x_2 is expressed entirely in terms of x_0. We now replace the x_2 on the rightt hand side of (4.10) with the expression on the right hand side of (4.16) to express x_3 directly in terms of x_0. Repeating this process $n - 1$ times we can express all the x_i in terms of x_0.

Example 4.1. Construct a numerical solution of the differential equation

$$\frac{dx}{dt} = x^2 t \ , \ \forall \ t \in [0.1].$$

such that

$$x(0) = 1.$$

on the interval $[0, 1]$.

Let's set $n = 10$, and let

$$\begin{aligned}
\Delta t &= \frac{1-0}{n} = .\frac{1}{10} \\
t_0 &= 0 \\
t_1 &= t_0 + \Delta t = 0.1 \\
t_2 &= t_0 + 2\Delta t = 0.2 \\
&\vdots \\
t_{10} &= t_0 + 10\Delta t = 1
\end{aligned}$$

and let x_i, $i = 0, \ldots, 10$ represent the values of $x(t)$ when $t = 0, \ldots, 10$. Since in this example

$$F(t, x) = x^2 t$$

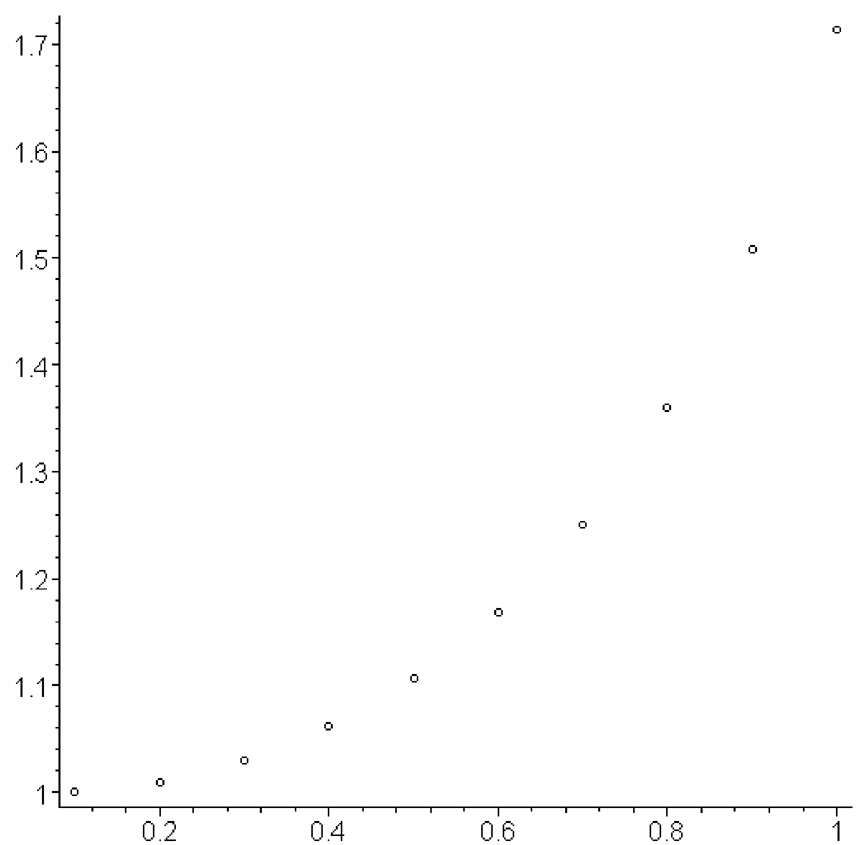
equations (4.8) - (4.14) take the form

$$\begin{aligned}
x_1 &= x_0 + \Delta t t_0 x_0^2 \\
x_2 &= x_1 + \Delta t t_1 x_1^2 \\
x_3 &= x_2 + \Delta t t_2 x_2^2 \\
&\vdots \\
x_{10} &= x_9 + \Delta t t_9 x_9^2
\end{aligned}$$

Since $\Delta t = \frac{1}{10}$, $t_i = \frac{i}{10}$ and $x_0 = x(0) = 1$, in this example, these equations can also be written as

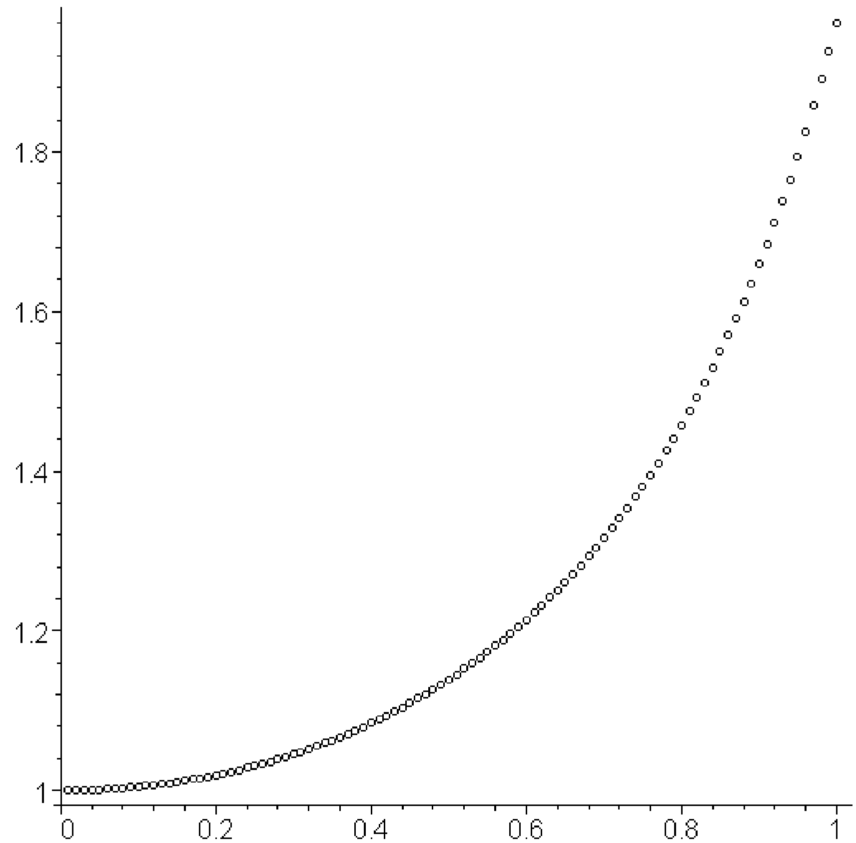
$$\begin{aligned} x_1 &= 1 \\ x_2 &= x_1 + \frac{0.1}{10}x_1^2 = 1 + (0.01)(1) = 1.01 \\ x_3 &= x_2 + \frac{0.2}{10}x_2^2 = (1.01) + (0.02)(1.01) = 1.0302 \\ &\vdots \\ x_{10} &= x_9 + \frac{0.9}{10}x_9^2 = 1.712852586 \end{aligned}$$

The data given above, of course, is not so useful in getting a feel for our solution $x(t)$ of the differential equation. To gain a little more intuition as to what our solution looks like, we can plot the pairs of points (t_i, x_i), $i = 0, 1, \ldots, 10$ in the tx plane. Such a plot is given below.



By mentally connecting the dots, we can get an idea of what the graph of our solution looks like.

Alternatively, we can choose our number of sample points n to very large, say $n = 100$, repeat the calculation (on a computer) and plot the results. Doing so we get a graph like



which is not only far more accurate (in matching the exact solution), but also contains so many data points that we don't even have to imagine connecting them to see the graph of $x(t)$.

Below I give a simple Maple routine that automated this calculation:

(1) n:= 100;
(2) x[0] := 1.0;
(3) f := (x,t) -> t*x^2;
(4) dt := (1-0)/n;
(5) for i from 0 to n do t[i] := i*d: od:
(6) for j from 1 to n do x[j] := x[j-1] + dt*f(x[j-1],t[j-1]): od:

In the first line I declare the number of sample points to be 1000.

In the second line I declare the initial value of x to be 1.0

In the third line I declare the function appearing on the right hand side of the differential equation.

In the fourth line I declare the value for interval Δt between adjacent sample points.

In the fifth line I create values for all the points $t_i = i * \Delta t$.

In the sixth line I recursively apply the difference relation (4.7) to calculate all the $x_i = x(t_i)$.

To see a plot of these points you can use the following Maple commands

(1) with(plots);
(2) pointlist := {seq([t[n],x[n]],n=0..100)}:
(3) pointplot(pointlist);