LECTURE 8

Autonomous Systems and Stability

An *autonomous system* is a system of ordinary differential equations of the form

$$\begin{aligned}\frac{dx_1}{dt} &= F_1(x_1,\ldots,x_n)\\ \frac{dx_2}{dt} &= F_2(x_1,\ldots,x_n)\\ &\vdots\\ \frac{dx_n}{dt} &= F_n(x_1,\ldots,x_n)\end{aligned}$$

or, in vector notation,

$$\mathbf{x}' = \mathbf{F}(\mathbf{x})$$

That is to say, an autonomous system is a system of ODEs in which the underlying variable t does not appear explicitly in the defining equations.

For example, homogeneous linear systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is a constant matrix are autonomous (with $\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x}$).

What's especially nice about autonomous linear systems is that the associated direction field plots are independent of t, and so one can get a fairly good idea of what the solutions look like by staring the direction field associated with the vector-valued function $\mathbf{F}(\mathbf{x})$.

In this lecture we shall focus on the behavior of solutions near *critical points*.

DEFINITION 8.1. *A point $\mathbf{x}_0$ is a* ***critical point*** *for an autonomous system*

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \tag{1}$$

if $\mathbf{F}(\mathbf{x}_0) = \mathbf{0}$.

The first thing to point out is that if $\mathbf{x}_0$ is a critical point of (1) then the constant function

$$\mathbf{x}(t) = \mathbf{x}_0$$

is always a solution of the differential equation: for

$$0 = \frac{d\mathbf{x}_0}{dt} = \frac{d\mathbf{x}}{dt}(t) = \mathbf{F}(\mathbf{x}(t)) = \mathbf{F}(\mathbf{x}_0) = 0$$

DEFINITION 8.2. *A critical point $\mathbf{x}_0$ of an autonomous system is said to be* ***stable*** *if the following condition holds:*

- *For any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\mathbf{\Phi}(t)$ is a solution of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ satisfying*

  $$\|\mathbf{\Phi}(0) - \mathbf{x}_0\| < \delta$$

  then for all $t > 0$, $\mathbf{\Phi}(t)$ exists (as a solution) and satisfies

  $$\|\mathbf{\Phi}(t) - \mathbf{x}_0\| < \varepsilon \quad .$$

That is, is to say that if $\mathbf{x}_0$ is a critical point, then if a solution "starts off close to $\mathbf{x}_0$", then it stays close to $\mathbf{x}_0$ for all positive t.[1]

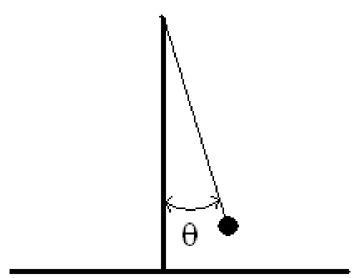
A related but distinct concept is that of an **asymptotically stable** critical point:

Definition 8.3. *A critical point $\mathbf{x}_0$ of an autonomous system is said to be* ***asymptotically stable*** *if the following condition holds:*

- *There exists a $\delta_0 > 0$ such that if $\mathbf{\Phi}(t)$ is a solution of $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ satisfying*
$$\|\mathbf{\Phi}(0) - \mathbf{x}_0\| < \delta_0$$
then
$$\lim_{t\to\infty} \mathbf{\Phi}(t) = \mathbf{x}_0 \quad .$$

Asymptotic stability is a bit stronger than mere stability; because the stability just requires that solutions that start near a critical point never stray far from that critical point, asymptotic stability, on the other hand, requires that solutions starting near the critical point $\mathbf{x}_0$ have to eventually (well, in the limit $t = \infty$ anyway) settle in at $\mathbf{x}_0$.

Here is a simple physical situation that distinguishes between asymptotically stable and stable critical points. Consider a pendulum.



One possible motion is the pendulum just resting at $\theta = 0$. This corresponds to the solution at a critical point. If you displace the pendulum by a small angle δ and then let go, there are two possibilities.

In the frictionless case, the pendulum just rocks back and forth indefinitely between $\theta = -\delta$ and $\theta = \delta$, but always with $|\theta(t) - 0| \leq \delta$. In this situation $\theta = 0$ is a stable critical point but not an asymptotically stable critical point (as the pendulum continues to rock back and forth forever).

In the (more realistic) case where there is friction at play, then eventually the rocking motions die down to the steady rest position. So for a damped pendulum (meaning a pendulum with friction acting) $\theta = 0$ is an asymptotically stable critical point.

Let's now look at the example of a pendulum a bit more quantitatively. The rotational motion analog of Newton's second law is

$$\textit{torque} = (\textit{rotational inertia})\,(\textit{angular acceleration})$$

[1]This is the geometric interpretation given in the text; but it's not quite accurate. What the ε, δ test really says is that if $\mathbf{x}_0$ is a critical point, by forcing solutions to pass through a small enough δ-neighborhood around $\mathbf{x}_0$, you can always ensure that they never stray outside an ε-neighborhood of $\mathbf{x}_0$.

or

$$\frac{d^2\theta}{dt^2} = \frac{1}{mL^2}\left[-mgL\sin\theta - cL\frac{d\theta}{dt}\right]$$

Here mL^2 is the moment of inertia of a mass m displaced from the pivot point by a distance L (the length of the pendulum string). The term $-mgL\sin\theta = \mathbf{r}\times\mathbf{F}_g$ (or at least the relevant component on the right hand side). The term $-cL\frac{d\theta}{dt}$ represents the force of friction. We can rewrite this equation as

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\sin\theta = 0$$

where

$$\gamma = \frac{c}{mL}$$
$$\omega = \sqrt{\frac{g}{L}}$$

To view this second order differential equation as an autonomous system of first order differential equations we set

$$x_1 = \theta$$
$$x_2 = \frac{d\theta}{dt}$$

to get

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = -\omega^2\sin(x_1) - \gamma x_2$$

Thus, for this system

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x_2 \\ -\omega^2\sin(x_1) - \gamma x_2 \end{bmatrix}$$

The critical points are then the solution of $\mathbf{F}(\mathbf{x}) = \mathbf{0}$

$$\Longrightarrow \left\{\begin{array}{r} x_2 = 0 \\ -\omega^2\sin(x_1) - \gamma x_2 = 0 \end{array}\right. \Longrightarrow \left\{\begin{array}{l} x_2 = 0 \\ x_1 = k\pi \end{array}\right. , \quad k\in\mathbb{Z}$$

Now $x_2 = 0 \implies \frac{d\theta}{dt} = 0$, meaning at these critical points the pendulum is at rest. The critical points where $x_1 = 0, \pm2\pi, \pm4\pi, \ldots$ correspond to the pendulum resting at the bottom of its swing; these are asymptotically stable critical points (if $c \neq 0$). The critical points where $x_1 = \pm\pi, \pm3\pi, \pm5\pi, \ldots$ correspond to the pendulum resting at the top of its swing. These are the unstable critical points.

1. Stability of Linear Autonomous Systems

We have already discussed the solutions of autonomous systems of the form

$$\mathbf{x}' = \mathbf{A}\mathbf{x} \tag{2}$$

Note that so long as $\mathbf{A}$ is a nonsingular matrix (i.e. so long as the only solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$), we will have only one critical point, namely $\mathbf{x}_0 = \mathbf{0}$.

When we were examining the solutions of linear systems of the form (2),we found were three basic cases for 2×2 systems. We'll now look at these basic cases again to assess their stability around the critical point $\mathbf{x}_0 = \mathbf{0}$.

- If $\mathbf{A}$ has two real eigenvalues r_1, r_2 and two corresponding eigenvectors $\xi^{(1)}$, $\xi^{(2)}$, the general solution of (2) took the form

$$\mathbf{x}(t) = c_1 e^{r_1 t}\xi^{(1)} + c_2 e^{r_2 t}\xi^{(2)}$$

Evidently, such solutions will be stable and asymptotically stable at $\mathbf{0}$ whenever r_1 and r_2 are both negative. If, however, one or both of the eigenvalues are positive then $\mathbf{x}(t)$ will be unstable as

$$\lim_{t\to\infty} \|\mathbf{x}(t)\| \to \infty$$

in these cases.

- If $\mathbf{A}$ has a pair of complex conjugate eigenvalues $\lambda_\pm = r \pm i\mu_1$ and a corresponding pair of complex conjugate eigenvectors $\mathbf{v}, \mathbf{v}^*$. then the general solution of (2) can be written as

$$\mathbf{x}(t) = c_1 \left(e^{rt}\cos(\mu t) - e^{rt}\sin(\mu t)\right)\xi + c_2 \left(e^{rt}\sin(\mu t) + e^{rt}\cos(\mu t)\right)\eta$$

where

$$\xi = \mathrm{Re}(\mathbf{v}) = \frac{1}{2}(\mathbf{v} + \mathbf{v}^*)$$
$$\eta = \mathrm{Im}(\mathbf{v}) = \frac{1}{2i}(\mathbf{v} - \mathbf{v}^*)$$

and so solutions will be stable and asymptotically stable if $r < 0$, stable if $r = 0$, and unstable if $r > 0$.

- If $\mathbf{A}$ is non-diagonalizable with a single eigenvalue r, then the solution of (2) will look like

$$\mathbf{x}(t) = c_1 e^{rt}\xi + c_2 \left(te^{rt}\xi + e^{rt}\eta\right)$$

where ξ is the eigenvector of $\mathbf{A}$ corresponding to the eigenvalue r and η is the solution of $(\mathbf{A} - r\mathbf{I})\eta = \xi$. Again, it is the exponential factor e^{rt} that dictates the behavior of solutions near $\mathbf{0}$ for positive t; $\mathbf{x}_0 = \mathbf{0}$ will be stable and asymptotically stable if r is negative, unstable if r is postive. When $r = 0$, the factor of t in the second solution will destabilize the general solution.

Despite the different function form of the solutions $\mathbf{x}(t)$ for these situations, the discussion above, shows the stability properties of $(0,0)$ can be inferred directly from the eigenvalues of the matrix $\mathbf{A}$. These characterizations I summarize below:

two real eigenvalues r_1, r_2

eigenvalue property	type of critical point	stability
$r_1 > r_2 > 0$	node	unstable
$r_1 < r_2 < 0$	node	stable
$r_1 < 0 < r_2$	saddle point	unstable

one real eigenvalues r

eigenvalue property	type of critical point	stability
$r > 0$	proper or improper node	unstable
$r < 0$	proper or improper node	stable

two complex eigenvalues $\alpha \pm i\beta$

eigenvalue property	type of critical point	stability
$\alpha > 0$	spiral point	unstable
$\alpha < 0$	spiral point	asymptotically stable
$\alpha = 0$	center	stable

2. Determination of Trajectories

For two dimensional autonomous systems the trajectories of solutions can sometimes be found by eliminating the appearance of the underlying variable t from the system of differential equations - via the identity

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad .$$

Indeed, if

$$\begin{aligned} \frac{dx}{dt} &= F(x,y) \\ \frac{dy}{dt} &= G(x,y) \end{aligned}$$

then x and y are related by the following first order differential equation

(3) $$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)}$$

Below we give some examples, we find the trajectories of an autonomous system by solving (3).

Example 8.4. Find the critical points and trajectories of the system

$$\begin{aligned} \frac{dx}{dt} &= 6 - 3y \\ \frac{dy}{dt} &= -12 + 3x^2 \end{aligned}$$

We have

$$\mathbf{0} = \mathbf{F}(x,y) = \begin{bmatrix} 4-2y \\ -12+3x^2 \end{bmatrix} \quad \Longrightarrow \quad \begin{cases} 4-2y = 0 \\ -12+3x^2 = 0 \end{cases}$$
$$\Longrightarrow \quad \begin{cases} y = 2 \\ x = \pm 2 \end{cases}$$

Thus, the critical points are $(2,2)$ and $(-2,2)$.

Next we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-12+3x^2}{6-3y}$$

or

(*) $$-3x^2 + 12 + (6-3y)\frac{dy}{dx} = 0$$

This ODE is separable and so we can solve it like

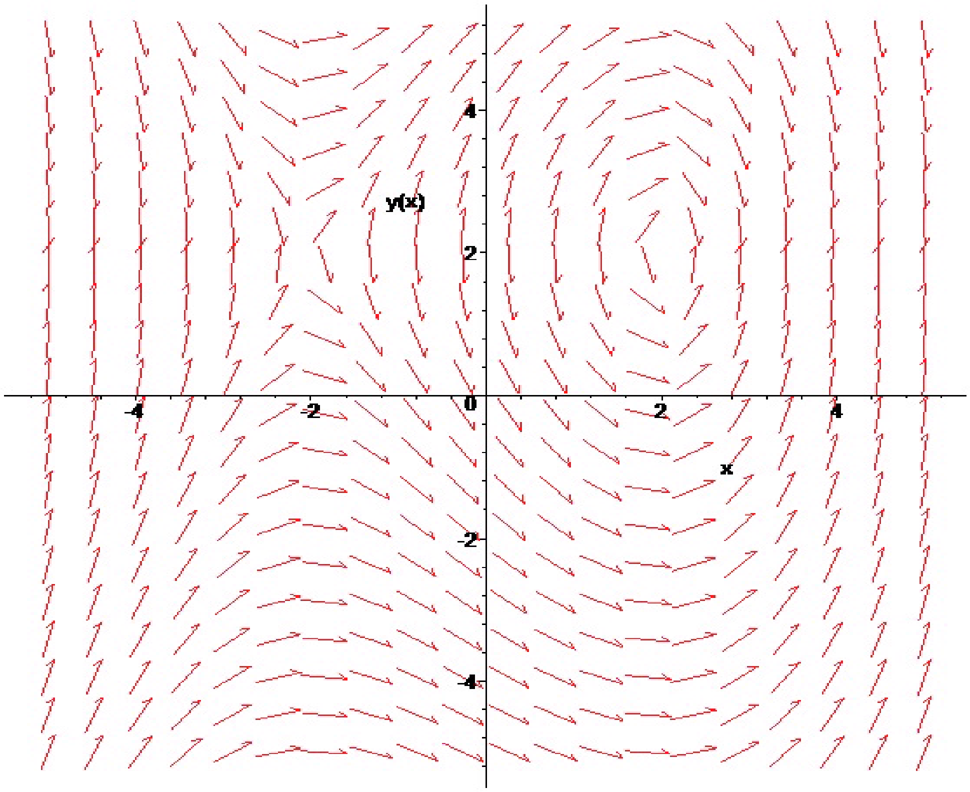
$$\int \left(-3x^2+12\right) dx + \int (6-3y)\, dy = C$$

$$\begin{aligned} &\Longrightarrow \quad -x^3 + 12x + 6y - \frac{3}{2}y^2 = C \\ &\Rightarrow \quad y^2 - 4y + \left(\frac{2}{3}x^3 - 8x + C'\right) = 0 \end{aligned}$$

or

$$y(x) = \frac{4 \pm \sqrt{16 - 4\left(\left(\frac{2}{3}x^3 - 8x + C'\right)\right)}}{2}$$

Below is a plot of some of these tranjectories (actually it's the direction field plot for the ODE (*) above)



8-4.jpg

Note that there appears to be a stable point around (-2,2).

3. Stability of Locally Linear Systems

An autonomous system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

is said to be **locally linear** around a critical point $\mathbf{x}_0$ if the vector function on the right hand side be expressed as

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}(\mathbf{x} - \mathbf{x}_0) + \mathbf{g}(\mathbf{x}) \tag{2}$$

with

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|\mathbf{g}(\mathbf{x})\|}{\|\mathbf{x} - \mathbf{x}_o\|} = 0 \tag{3}$$

It turns out that the criterion is satisfied automatically, so long as the component functions of $\mathbf{F}(\mathbf{x})$ have continuous partial derivatives up to order two at $\mathbf{x}_0$. This because in this situation, the expression (2) for $\mathbf{F}(\mathbf{x})$ can be regarded as the separation of the terms of order ≥ 2 from the linear terms in the Taylor expansion of $\mathbf{F}(\mathbf{x})$ about $\mathbf{x}_0$. Indeed, using the Taylor theorem for functions of two variables is how we will identify the corresponding linearizng approximation to a locally linear system.

Let's consider a two component system in explicit form

$$\frac{dx}{dt} = F(x,y) = F(x_0,y_0) + \frac{\partial F}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial F}{\partial y}(x_0,y_0)(y-y_0) + \mathcal{O}\left((x-x_0)^2,(y-y_0)^2\right)$$
$$\frac{dy}{dt} = G(x,y) = G(x_0,y_0) + \frac{\partial G}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial\backslash G}{\partial y}(x_0,y_0)(y-y_0) + \mathcal{O}\left((x-x_0)^2,(y-y_0)^2\right)$$

At a critical point,

$$\mathbf{0}=\mathbf{F}(x_0,y_0)=\left[\begin{array}{c}F(x_0,y_0)\\G(x_0,y_0)\end{array}\right]\quad\Rightarrow\quad F(x_0,y_0)=0=G(x_0,y_0)$$

So the first terms of the Taylor expansions drop out. If we then set

$$\mathbf{u}(t)=\mathbf{x}(t)-\mathbf{x}_0$$

Then $\mathbf{u}(t)$ will satisfy

$$\frac{d\mathbf{u}}{dt}=\left(\begin{array}{cc}\frac{\partial F}{\partial x}(x_0,y_0)&\frac{\partial F}{\partial y}(x_0,y_0)\\\frac{\partial G}{\partial x}(x_0,y_0)&\frac{\partial F}{\partial y}(x_0,y_0)\end{array}\right)\left(\begin{array}{c}u_1(t)\\u_2(4)\end{array}\right)+\mathcal{O}\left(\|\mathbf{u}\|^2\right)$$

Ignoring the $\mathcal{O}\left(\|\mathbf{u}\|^2\right)$ term, we see can study the behavior of the solutions $\mathbf{x}(t)$ near $\mathbf{x}_0$, by analyzing the solutions of the simple linear system

$$\frac{d\mathbf{u}}{dt}=\mathbf{A}\mathbf{u}$$

near $\mathbf{u}_0=\mathbf{0}$. Here, of course, $\mathbf{A}$ is the constant matrix

$$\mathbf{A}=\left(\begin{array}{cc}\frac{\partial F}{\partial x}(x_0,y_0)&\frac{\partial F}{\partial y}(x_0,y_0)\\\frac{\partial G}{\partial x}(x_0,y_0)&\frac{\partial F}{\partial y}(x_0,y_0)\end{array}\right)$$

The stability of critical points for a locally linear system and thus be decided by computing the matrix $\mathbf{A}$ from $\mathbf{F}(\mathbf{x})$ and examining the eigenvalues of $\mathbf{A}$.

3.1. Example 8.4 done analytically.

Recall the nonlinear system of Example 8.4

$$\frac{d}{dt}\left[\begin{array}{c}x\\y\end{array}\right]=\left[\begin{array}{c}6-3y\\-12+3x^2\end{array}\right]$$

and that this system had critical points at $(-2,2)$ and $(2,2)$. This system is locally linear at both these critical points because both components of $\mathbf{F}(x,y)$ are polynomials in x and y (and so have continuous partial derivatives.

3.1.1. $\mathbf{x}_0=(2,2)$.

Using the Taylor formula

$$f(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(x-y_0)+\mathcal{O}\left(\|\mathbf{x}-\mathbf{x}_0\|^2\right)$$

we have

$$\begin{aligned}4-2y&=0+0(x-2)+(-3)(y-2)+\mathcal{O}\left(\|\mathbf{x}-\mathbf{x}_0\|^2\right)\\12-3x^2&=0+(12)(x-2)+0(y-2)+\mathcal{O}\left(\|\mathbf{x}-\mathbf{x}_0\|^2\right)\end{aligned}$$

Taylor expand $\mathbf{F}(x,y)$ about $\mathbf{x}_0=(2,2)$ yields

$$\mathbf{F}(x,y)=\left[\begin{array}{cc}0&-3\\12&0\end{array}\right]\left[\begin{array}{c}x-2\\y-2\end{array}\right]+\mathcal{O}\left(\|\mathbf{x}-\mathbf{x}_0\|^2\right)$$

To analyze the nature of solutions in the vicinity of $\mathbf{x}_0=(2,2)$ we make a change of variable

$$\begin{aligned}u(t)&=x(t)-2\\v(t)&=y(t)-2\end{aligned}$$

and study instead

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} &= \frac{d}{dt}\begin{bmatrix} x-2 \\ y-2 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} \\ &\approx \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}\begin{bmatrix} x-2 \\ y-2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

This is a simple linear system, where the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 0 & -3 \\ 12 & 0 \end{bmatrix}$$

We have

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 + 36 \quad \Rightarrow \quad \lambda = 6i$$

and so we have two pure imaginary eigenvalues. Thus, the linearized system has a *stable center* at $(0,0)$, which which is in agreement with our original graphical picture.

3.2. $\mathbf{x}_0 = (-2,2)$. Let's now repeat this analysis for the second critical point. In this case, the linearization of $\mathbf{F}(x,y)$ is

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}$$

where

$$\mathbf{A} = \left(\begin{array}{cc} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{array}\right)\Bigg|_{\substack{x=-2 \\ y=2}} = \left(\begin{array}{cc} 0 & -3 \\ 6x & 0 \end{array}\right)\Bigg|_{\substack{x=-2 \\ y=2}} = \left(\begin{array}{cc} 0 & -3 \\ -12 & 0 \end{array}\right)$$

and thus

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 36 \quad \Rightarrow \quad \lambda = \pm 6$$

Since $\mathbf{A}$ has a positive eigenvalue, we conclude that the linearization of $\mathbf{x}(t)$ near $(-2,2)$ is *unstable*, hence $(-2,2)$ is an unstable critical point for the original linear system. (This conclusion is also in agreement with our original graphical picture.)