LECTURE 10

Liapunov's Second Method

In the last lecture we analyzed the behavior of solutions near critical points by studying the solutions of an approximate linear system near the critical point. There are at least two situations where this kind of analysis is very tenuous: (i) when the critical point is at the origin and (ii) when the eigenvalues of the linear system are pure imaginary. In these cases the crux lies in the fact that a small, even infinitesimal change in the eigenvalues of the linear system can have a drastic effect on the nature of the solutions near critical point. For the asymptotic stability of a critical point requires these eigenvalues have strictly negative real part. If the real parts of these eigenvalues are zero, then a small alteration in the eigenvalues (brought on by the contribution of the non-linear terms of the original nonlinear system) can cause the solutions to either shoot off to infinity (if the alteration is positive) or converge asymptotically to the critical point (if the alteration is negative).

In this lecture we will develop an alternative means for studying the stability of solutions near critical points.

1. Prototype for Liapunov's Second Method

Liapunov's Second Method is a generalization to the theory of nonlinear systems of ODEs of two basic physical principles:

- A *state* of a conservative physical system is stable only if its potential enery has a local minimum at that state.
- The total energy of a conservative physical system is constant during the evolution of the system.

Consider the differential equations governing an undamped pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

in first order form

(1)
$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -\frac{g}{L}\sin x\end{aligned}$$

where we have used the following change of variables.

$$\begin{aligned}x &:= \theta\\ y &:= \frac{d\theta}{dt}\end{aligned}$$

Evidently, the critical points of this system are

$$\left.\begin{aligned}y &= 0\\ -\tfrac{g}{L}\sin x &= 0\end{aligned}\right\} \quad\Longrightarrow\quad y = 0,\ x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

In particular, the system (1) has a critical point at the origin, and the linear system there has associated matrix

$$\left.\frac{d\mathbf{F}}{d\mathbf{x}}\right|_{(0,0)} := \left.\left[\begin{array}{cc} \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \\ \frac{\partial}{\partial x}\left(-\frac{g}{L}\sin x\right) & \frac{\partial}{\partial y}\left(-\frac{g}{L}\sin x\right)\end{array}\right]\right|_{(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ -\frac{g}{L} & 0\end{array}\right]$$

The eigenvalues of this matrix are

$$0 = \det\left(\left[\begin{array}{cc} 0-\lambda & 1 \\ -\frac{g}{L} & 0-\lambda\end{array}\right]\right) \quad \Longrightarrow \quad \lambda = \pm i\sqrt{\frac{g}{L}}$$

Since the eigenvalues are purely imaginary, our analysis by linearization is not applicable.

However, since there is no friction acting on the system, we know that the total energy of the system is constant. We have

$$\begin{aligned} \text{total energy} = & \ (\text{kinetic energy}) + (\text{potential energy}) \\ = & \ \frac{1}{2}mv^2 + mgh \end{aligned}$$

Since the velocity of the weight at the end of the pendulum is $L\frac{d\theta}{dt}$

$$KE = \frac{1}{2}mL^2y^2$$

And since the height of the pendulum weight is given by $h = L(1-\cos\theta)$ we have

$$PE = mgL(1-\cos\theta)$$

Thus, the total energy is

$$E = \frac{1}{2}mL^2y^2 + mgL(1-\cos x)$$

Since the energy is conserved

$$0 = \frac{dE}{dt} = mL^2y\frac{dy}{dt} + mgL\sin\theta\frac{dx}{dt} \tag{2}$$

Indeed, if we substitute the right hand sides of (1) into (2) we get

$$\frac{dE}{dt} = mL^2y\left(-\frac{g}{L}\sin x\right) + mgL\sin\theta(y) = 0$$

Now note that near the critical point $(0,0)$ where y, x are small

$$\begin{aligned} E &= \frac{1}{2}mL^2y^2 + mgL(1-\cos x) \approx \frac{1}{2}mL^2y^2 + mgL\left(1-\left(1-\frac{x^2}{2}+\cdots\right)\right) \\ &\approx \frac{1}{2}mL^2y^2 + \frac{1}{2}mgLx^2 \end{aligned}$$

The condition that E is constant thus requires x and y to be bound to an ellispe

$$\frac{x^2}{mgL} + \frac{y^2}{mL^2} = \frac{2E}{mgL^2}$$

From this we can infer that the trajectories that pass close to the critical point at $(0,0)$ can never stray to far from it. Thus, the critical point at $(0,0)$ is **stable** (but not necessarily assymptotically stable).

2. Positive/Negative Definite Functions

We'll now develop the two Liapunov theorems which generalize the mathematical phenonmenon underlying preceding analysis.

First of all, we need an appropriate replacement for the total energy function. It turns out that all we really need is a function with appropriate positivity properties near the critical point.

DEFINITION 10.1. *We say that a function Φ vanishing at a point $\mathbf{x}_o$ is* ***positive definite*** *about $\mathbf{x}_o$ if $\Phi(\mathbf{x}_o) = 0$ and for all $\mathbf{x} \neq \mathbf{x}_o$ within a sufficiently small neighborhood of $\mathbf{x}$. Φ is* ***positive semi-definite*** *about $\mathbf{x}_o$ if $\Phi(\mathbf{x}_o) = 0$ and $\Phi(\mathbf{x}) \geq 0$ for all $\mathbf{x}$ within a sufficiently small neighborhood of $\mathbf{x}_o$. Similarly, Φ is, respectively,* ***negative definite*** *or* ***negative semi-definite*** *about $\mathbf{x}_o$ if $\Phi(\mathbf{x}_o) = 0$ and, respectively, $\Phi(\mathbf{x}) < 0$ or $\Phi(\mathbf{x}) \leq 0$ for all $\mathbf{x} \neq \mathbf{x}_o$ in a small neighborhood about $\mathbf{x}_o$.*

In other words, Φ is positive definite about $\mathbf{x}_o$ if $\Phi(\mathbf{x}_o) = 0$ and $\mathbf{x}_o$ is a local minimum of Φ, and Φ is negative definite about $\mathbf{x}_o$ if $\Phi(\mathbf{x}_o) = 0$ and $\mathbf{x}_o$ is a local maximum.

The following theorem is proved in Math 4023 (Vector Calculus):

THEOREM 10.2. *Let Φ be a continuous function of two variables with continuous partial derivatives on a domain D in the plane up to degree 3. A point $\mathbf{x}_o = (x_0, y_0)$ is a local minimum of Φ provided the following three conditions hold:*

(i) $\left.\frac{\partial \Phi}{\partial x}\right|_{\mathbf{x}_o} = 0 = \left.\frac{\partial \Phi}{\partial y}\right|_{\mathbf{x}_0}$

(ii) $\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_{\mathbf{x}_o} > 0$

(iii) $\left.\left(\frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2} - \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)^2\right)\right|_{\mathbf{x}_o} > 0$

If, on the other hand, (i) and (iii) hold and $\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_{\mathbf{x}_o} < 0$, the $\mathbf{x}_o$ is a local maximum.

EXAMPLE 10.3. Consider the function

$$\Phi(x, y) = ax^2 + bxy + cy^2$$

We have

$$\begin{aligned}
\left.\frac{\partial \Phi}{\partial x}\right|_{(0,0)} &= \left.(2ax + by)\right|_{(0,0)} = 0 \\
\left.\frac{\partial \Phi}{\partial y}\right|_{(0,0)} &= \left.(bx + 2cy)\right|_{(0,0)} = 0 \\
\left.\frac{\partial^2 \Phi}{\partial x^2}\right|_{(0,0)} &= 2a \\
\left.\left(\frac{\partial^2 \Phi}{\partial x^2}\frac{\partial^2 \Phi}{\partial y^2} - \left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)^2\right)\right|_{(0,0)} &= 4ac - b^2
\end{aligned}$$

and $(0,0)$ is a local minimum of Φ if

$$a > 0 \quad \text{and} \quad 4ac - b^2 > 0$$

3. The Liapunov Theorems

The basic idea underlying the Liapunov theorems is as follows. Let **c** be a local minimum of some function $\phi(\mathbf{x})$ and let $\gamma(t) = (\gamma_x(t), \gamma_y(t))$ be any curve that passes closely by **c**. Then the rate of increase of ϕ as you move along $\gamma(t)$ is, by the chain rule for functions of several variables,

$$\frac{d\phi}{dt} = \frac{d}{dt}\phi(\gamma(t)) = \frac{\partial\phi}{\partial x}\frac{d\gamma_x}{dt} + \frac{\partial\phi}{dy}\frac{d\gamma_y}{dt}$$

Suppose you observe that $\frac{d\phi}{dt}$ is positive whenever $\gamma(t)$ is very close to the minimum of ϕ. You can infer that the curve $\gamma(t)$ must be moving away from the minimum of ϕ. If, on the other hand, $\frac{d\phi}{dt}$ is always negative when $\gamma(t)$ is close to **c**, you can infer that the curve $\gamma(t)$ is moving *towards* the minimum of ϕ.

Now suppose you have a function $\phi(\mathbf{x})$ that is has a local maximum at the critical point **c** of an autonomous system

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) \quad . \tag{3}$$

Choose a curve $\gamma(t) = \mathbf{x}(t) = (x(t), y(t))$ corresponding to a solution of (3) passing close to the critical point **c**. Then

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{dy}\frac{dy}{dt} = \frac{\partial\phi}{\partial x}F_x + \frac{\partial\phi}{\partial y}F_y \equiv \nabla\phi\cdot\mathbf{F}$$

and so, by the same reasoning as above, you can infer that if $\nabla\phi\cdot\mathbf{F}$ is negative whenever **x** is close to a critical point **c**, that the solution trajectories of (3) near **c** must be moving towards the critical point **c**. And so if $\nabla\phi\cdot\mathbf{F}$ is negative whenever **x** is close to a critical point **c**, **c** will be an asymptotically stable critical point.

This idea is the basis for Liapunov's theorems.

THEOREM 10.4. *Consider an autonomous system*

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \tag{4}$$

with a critical point at the origin. If there exists a function $\Phi(\mathbf{x})$ *that is*

- *continuous and has continous first partial derivatives;*
- $\Phi(\mathbf{x})$ *is positive definite about the origin;*

Then if

$$\frac{d\Phi}{dt}(\mathbf{x}) := \frac{\partial\Phi}{\partial x_1}\frac{\partial x_1}{dt} + \frac{\partial\Phi}{\partial x_2}\frac{\partial x_2}{dt} + \cdots + \frac{\partial\Phi}{\partial x_n}\frac{\partial x_n}{dt} = \nabla\Phi\cdot\mathbf{F}(\mathbf{x})$$

is negative semi-definite about the origin, the origin is a stable critical point for (4). If, in fact, $\frac{d\Phi}{dt}(\mathbf{x}) < 0$ *for all* **x** *within a small ball about the origin, then the origin is an asympotically stable critical point for (4).*

The second Liapunov theorem gives a criterion for instability.

THEOREM 10.5. *Let the origin be an isolated critical point for the autonomous systems (4), and let* $\Phi(\mathbf{x})$ *be a function that is continuous and has continuous first partial derivatives. Suppose* $\Phi(0,0) = 0$ *and every neighborhood of the origin has at least one point at which* Φ *is positive and that there is a domain* D *containing the origin such that* $\nabla\Phi\cdot\mathbf{F}(x)$ *is positive definite, then the origin is an unstable critical point.*

EXAMPLE 10.6. Show that the critical point of $(0,0)$ of the autonomous system

$$\begin{aligned}\frac{dx}{dt} &= -x - xy^2 \\ \frac{dy}{dt} &= -y - x^2y\end{aligned} \tag{5}$$

is asympotically stable.

- We have

$$\mathbf{F}(x,y) = \begin{bmatrix} -x - xy^2 \\ -y - x^2 y \end{bmatrix}$$

Set

$$\Phi(x,y) = ax^2 + bxy + cy^2$$

As shown above, $\Phi(x,y)$ is positive definite if and only if $a > 0$ and $4ac - b^2 > 0$. Now consider

$$\frac{d\Phi}{dt} := \nabla\Phi \cdot \mathbf{F} = [2ax + by, bx + 2cy] \begin{bmatrix} -x - xy^2 \\ -y - x^2 y \end{bmatrix} = (2ax + by)\left(-x - xy^2\right) + (bx + 2cy)\left(-y - x^2 y\right)$$

$$= -a\left(2x^2 + x^2y^2\right) - b\left(2yx + y^3x + x^3y\right) - c\left(2y^2 + x^2y^2\right)$$

We have

$$\frac{\partial}{\partial x}\left(\frac{d\Phi}{dt}\right) = 0 = \frac{\partial}{\partial y}\left(\frac{d\Phi}{dt}\right)$$

$$\left.\frac{\partial^2}{\partial x^2}\left(\frac{d\Phi}{dt}\right)\right|_{(0,0)} = -4a$$

$$\left.\left(\left(\frac{\partial^2}{\partial x^2}\frac{d\Phi}{dt}\right)\left(\frac{\partial^2}{\partial y^2}\frac{d\Phi}{dt}\right) - \left(\frac{\partial^2}{\partial x \partial y}\frac{d\Phi}{dt}\right)^2\right)\right|_{(0,0)} = (-4a)(-4c) - (-2b)^2 = 4\left(4ac - b^2\right)$$

Thus, if we choose the parameters a, b, c so that $a > 0$, $4ac - b^2 > 0$, then Φ will have a local minimum at $(0,0)$, while $\frac{d\Phi}{dt}$ will have a local maximum at $(0,0)$. Thus, Φ will be postive definite about $(0,0)$, while $\frac{d\Phi}{dt}$ will be negative definite about $(0,0)$. The first Liapunov theorem then tells us that $(0,0)$ is a asymptotically stable critical point for the system (5).