LECTURE 3

Orthogonal Functions

1. Orthogonal Bases

The appropriate setting for our discussion of orthogonal functions is that of linear algebra. So let me recall some relevant facts about finite dimensional vector spaces. Every vector in $\mathbb{R}^n$ can be represented as a sum of the form

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i \tag{3.1}$$

where $\mathbf{e}_i$ is the unit vector along the i^{th} coordinate axis. However, if $\{\mathbf{f}_i\}$ is any set of n linearly independent vectors, then we also have a unique representation of $\mathbf{v}$ as

$$\mathbf{v} = \sum_{i=1}^{n} c_i \mathbf{f}_i \quad . \tag{3.2}$$

It should be noted, however, that the vectors $\mathbf{f}_i$ need not be orthogonal nor need they have unit length for the expansion (3.2) to work. If, however,

$$\mathbf{f}_i \cdot \mathbf{f}_j = \delta_{ij} \quad , \tag{3.3}$$

that is, if the $\mathbf{f}_i$ are orthonormal, then the coefficients c_i can easily be computed. For if we take the dot product of $\mathbf{v}$ with a basis vector $\mathbf{f}_i$ we get

$$\mathbf{f}_i \cdot \mathbf{v} = \mathbf{f}_i \cdot \left(\sum_{j=1}^{n} c_j \mathbf{f}_j \right) = \sum_{j=1}^{n} c_j \left(\mathbf{f}_i \cdot \mathbf{f}_j \right) = \sum_{j=1}^{n} c_j \delta_{ij} = c_i ;$$

that is to say,

$$c_i = \mathbf{f}_i \cdot \mathbf{v} \quad . \tag{3.4}$$

2. Bases of Orthogonal Functions

The relevance of these remarks now comes from the observation that the set $C[\mathbb{R},\mathbb{R}]$ of continous real-valued functions on the real line is also a vector space; for the operations of addition and scalar multiplication of functions are well-defined:

$$\begin{aligned} [f+g](x) &= f(x) + g(x) \\ [c * f](x) &= cf(x) \quad \forall\, c \in \mathbb{R}. \end{aligned}$$

Note, however, that $C[\mathbb{R},\mathbb{R}]$ is an infinite dimensional vector space. Indeed, the Taylor expansion (2.16) of ϕ can be thought of as an expansion of ϕ with respect to basis of monomial functions $\{\mathbf{f}_{mn}(x,t) = x^m t^n\}$. Unfortunately, this basis is not orthonormal, at least not with respect to any obvious inner product. Fourier series are better examples, since they constitute expansions of functions with respect to an orthonormal basis of $C[\mathbb{R},\mathbb{R}]$.

THEOREM 3.1. ***(Fourier Theorem)*** *If f is a function such that $f(x)$ and $f'(x)$ that are piecewise continuous on the interval $[0, L] \subset \mathbb{R}$, then the series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where the coefficients a_n, b_n are determined by

$$\begin{aligned} a_n &= \tfrac{2}{L}\int_0^L \cos\left(\tfrac{n\pi x'}{L}\right) f(x')dx' \\ b_n &= \tfrac{2}{L}\int_0^L \sin\left(\tfrac{2\pi x'}{L}\right) f(x')dx' \end{aligned}$$

converges pointwise to $f(x)$ for all $x \in [0, L]$.

In view of the formulas

$$\begin{aligned} \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) x &= \delta_{m,n} \\ \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) x &= 0 \\ \frac{2}{L}\int_0^L \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) x &= \delta_{m,n} \end{aligned}$$

the Fourier expansion of f amounts to a expansion of f with respect to the basis $\left\{\sin\left(\frac{m\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\}$ which is orthonormal with respect to the inner product

$$f \cdot g = \frac{2}{L}\int_0^L f(x)g(x)dx \quad .$$

Sturm-Liouville theory is a generalization of Fourier theory. It provides a means of constructing other sets of orthonomal bases for spaces of functions.

THEOREM 3.2. ***(Sturm-Liouville Theorem)*** *Consider a boundary value problem of the form*

$$\begin{aligned} \tfrac{d}{dx}\left(p(x)\tfrac{dy}{dx}\right) + \left(q(x) + \lambda r(x)\right) y &= 0 \\ c_1 y(a) + c_2 y'(a) &= 0 \\ d_1 y(b) + d_2 y'(b) &= 0 \end{aligned} \tag{3.5}$$

where $p(x)$ and $r(x)$ are smooth positive functions on the interval (a, b). Then

(i) *For all but a discrete set S of choices of λ, there are no solutions to (3.5). There exists a minimal $\lambda \in S$. and one can arrange the set S of admissible λ so that*

$$S = \{\lambda_0, \lambda_1, \lambda_2, \dots, \}$$

with

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

One then has

$$\lim_{n\to\infty} \lambda_n = +\infty \quad .$$

(ii) *To each eigenvalue $\lambda_n \in S$, there corresponds exactly one solution $\phi_{\lambda_n}(x)$ of (3.5).*

(iii) *If $\lambda_n, \lambda_m \in S$ and $\phi_{\lambda_n}(x)$, $\phi_{\lambda_m}(x)$ are the corresponding solutions, then*

$$\lambda_n \neq \lambda_m \qquad \Rightarrow \qquad \int_a^b \phi_{\lambda_n}(x)\phi_{\lambda_m}(x) r(x) dx = 0 \quad .$$

(iv) *The set of functions*

$$\left\{ \beta_m = \frac{\phi_{\lambda_m}(x)}{\left[\int_a^b \left(\phi_{\lambda_m}(x)\right)^2 r(x)ds\right]^{1/2}} \quad \middle| \quad m = 0, 1, 2, \cdots \right\}$$

form a complete orthonormal basis for space of piecewise continuous functions on the interval $[a, b]$*. That is to say, every piecewise continous function* $f : [a, b] \to \mathbb{R}$*, can be expanded as*

$$f(x) \approx \sum_{n=1}^{\infty} \alpha_n \beta_n(t) \tag{3.6}$$

where the coefficients α_n *are determined by*

$$\alpha_n = \int_a^b f(x)\beta_n(x)r(x)dx \quad . \tag{3.7}$$

The series (3.6) converges in the sense that

$$\lim_{N\to\infty} \int_a^b \left(f(x) - \sum_{n=1}^{N} \alpha_n \beta_n(x) \right)^2 dx \quad = \quad 0 \quad . \tag{3.8}$$

Remarks:

(i) Existence of Eigenvalues and Eigenfunctions. Statement (i) is made a plausible by considering the simple example

$$\left.\begin{array}{rcl} y'' + \lambda^2 y & = & 0 \\ y(0) & = & 0 \\ y(L) & = & 0 \end{array}\right\} \qquad \Rightarrow \qquad \left\{ \begin{array}{rcl} \lambda & = & \frac{n\pi}{L} \\ y(x) & = & A \sin\left(\frac{n\pi x}{L}\right) \end{array}\right.$$

and indeed in applications before one can make use of the expansion (3.6), one has to first find the eigenvalues λ_i and so part (i) may be regarded as proved constructively. (However, an abstract proof also exists.)

(ii) Uniqueness of Eigenfunctions. Statement (ii) follows from the existence and uniqueness theorem for second order linear ODE's.

(iii) Orthogonality Property of Eigenfunctions. Let ϕ_{λ_1} and ϕ_{λ_2} be solutions of (3.5) for $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. Assume $\lambda_1 \neq \lambda_2$. Then

$$\begin{aligned} \frac{d}{dx}\left(p(x)\phi'_{\lambda_1}\right) + \left(q(x) + \lambda_1 r(x)\right)\phi_{\lambda_1} &= 0 \\ \frac{d}{dx}\left(p(x)\phi'_{\lambda_2}\right) + \left(q(x) + \lambda_2 r(x)\right)\phi_{\lambda_2} &= 0 \end{aligned}$$

Multiplying the first equation by $\phi_{\lambda_2}$and the second by ϕ_{λ_1} and subtracting the two equations yields

$$\left(\lambda_1 - \lambda_2\right) r(x)\phi_{\lambda_1}\phi_{\lambda_2} = \phi_{\lambda_2}\frac{d}{dx}\left(p(x)\phi'_{\lambda_1}\right) - \phi_{\lambda_1}\frac{d}{dx}\left(p(x)\phi'_{\lambda_2}\right)$$

Integrating both sides of the expression above between $x = a$ and $x = b$ yields

$$
\begin{aligned}
(\lambda_1 - \lambda_2)\int_a^b \phi_{\lambda_1}(x)\phi_{\lambda_2}(x)r(x)dx &= \int_a^b \phi_{\lambda_2}(x)\frac{d}{dx}\left(p(x)\phi'_{\lambda_1}(x)\right)dx \\
&\quad - \int_a^b \phi_{\lambda_1}(x)\frac{d}{dx}\left(p(x)\phi'_{\lambda_2}(x)\right)dx \\
&= \left(\phi'_{\lambda_1}(x)\phi_{\lambda_2}(x)p(x)\right)\Big|_a^b - \int_a^b \phi'_{\lambda_1}(x)\phi'_{\lambda_2}(x)p(x)dx \\
&\quad - \left(\phi_{\lambda_1}(x)\phi'_{\lambda_2}(x)p(x)\right)\Big|_a^b + \int_a^b \phi'_{\lambda_1}(x)\phi'_{\lambda_2}(x)p(x)dx \\
&= r(b)\left(\phi'_{\lambda_1}(b)\phi_{\lambda_2}(b) - \phi_{\lambda_1}(b)\phi'_{\lambda_2}(b)\right) \\
&\quad -r(a)\left(\phi'_{\lambda_1}(a)\phi_{\lambda_2}(a) - \phi_{\lambda_1}(a)\phi'_{\lambda_2}(a)\right)
\end{aligned}
$$

Now, in order for the boundary conditions at $x = a$

$$
\begin{aligned}
c_1\phi_{\lambda_1}(a) + c_2\phi'_{\lambda_1}(a) &= 0 \\
c_1\phi_{\lambda_2}(a) + c_2\phi'_{\lambda_2}(a) &= 0
\end{aligned}
$$

to have solutions, with c_1, c_2 not both zero, we must have

$$\phi'_{\lambda_1}(a)\phi_{\lambda_2}(a) - \phi_{\lambda_1}(a)\phi'_{\lambda_2}(a) = 0 \quad .$$

This can be seen as follows: Recall from linear algebra that if $\mathbf{M}$ is a $n \times n$ matrix and $\mathbf{v}$ is a n-dimensional vector, then $\mathbf{M}\,\mathbf{v} = \mathbf{0}$ has non-trivial solutions if and only if $det\ \mathbf{M} = 0$. The statement above then follows by considering

$$\mathbf{M} = \begin{pmatrix} \phi_{\lambda_1}(a) & \phi'_{\lambda_1}(a) \\ \phi_{\lambda_2}(a) & \phi'_{\lambda_2}(a) \end{pmatrix} \qquad , \quad \mathbf{v} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad .$$

Similarly, in order for the boundary conditions at $x = b$ to be satisfied for $c_{3,}c_4$ not both zero we must have

$$\phi'_{\lambda_1}(b)\phi_{\lambda_2}(b) - \phi_{\lambda_1}(b)\phi'_{\lambda_2}(b) = 0 \quad .$$

Thus, we have

$$(\lambda_1 - \lambda_2)\int_a^b \phi_{\lambda_1}(x)\phi_{\lambda_2}(x)r(x)dx = 0$$

So,

$$\int_a^b \phi_{\lambda_1}(x)\phi_{\lambda_2}(x)r(x)dx = 0$$

if $\lambda_1 \neq \lambda_2$. □

(iv) Completeness of Eigenfunctions The hard thing to understand is the remarkable completeness property expressed in statement (iv). The proof of this statement is not terribly difficult - however, it does require a moderate digression into the Calculus of Variations. At the end of the course, time-permitting, we will develop the Calculus of Variations, and then prove (iv) as a sample application.

3. Examples

EXAMPLE 3.3. **Fourier Sine Series**:

$$
\begin{aligned}
y'' + \lambda y &= 0 \\
y(0) &= 0 \\
y(L) &= 0
\end{aligned}
$$

This is a Sturm-Liouville type problem with $p(x) = r(x) = 1$, $q(x) = 0$. The general solution of the ODE is given by

$$y(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right)$$

however such solutions will satisfy the boundary conditions $y(0) = y(\pi) = 0$ if and only if $B = 0$ and

$$\sqrt{\lambda} = \frac{n\pi}{L} \quad , \quad n = 1, 2, 3, \ldots \quad .$$

The Sturm-Liouville inner product is

$$(f, g) = \int_0^L f(x)g(x)dx$$

and the functions

$$y_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

constitute a complete orthonormal basis for the set of piecewise continuous functions on the interval $(0, L)$.

EXAMPLE 3.4. **Fourier Cosine Series**:

$$\begin{aligned} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y'(L) &= 0 \end{aligned}$$

This is a Sturm-Liouville type problem with $p(x) = r(x) = 1$, $q(x) = 0$. The general solution of the ODE is given by

$$y(x) = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right)$$

however such solutions will satisfy the boundary conditions $y(0) = y(\pi) = 0$ if and only if $A = 0$ and

$$\sqrt{\lambda} = \frac{2n\pi}{L} \quad , \quad n = 1, 2, 3, \ldots \quad .$$

The Sturm-Liouville inner product is

$$(f, g) = \int_0^L f(x)g(x)dx$$

and the functions

$$y_n = \sqrt{\frac{2}{L}}\cos\left(\frac{2n\pi x}{L}\right)$$

constitute a complete orthonormal basis for the set of piecewise continuous functions on the interval $(0, L)$.

EXAMPLE 3.5. **Bessel Functions**

$$\frac{d}{dx}\left(x\frac{dJ_{n,a}}{dx}\right) - \frac{n^2}{x}J_{n,a} + \left(x + a^2\right)J_{n,a} = 0$$

EXAMPLE 3.6. **Legendre Functions**

$$\frac{d}{dx}\left(\left(1 - x^2\right)\frac{dL_l}{dx}\right) + \left(1 + l(l+1)\right)L_l = 0 \tag{3.9}$$

EXAMPLE 3.7. **Hermite Functions**

$$\frac{d}{dx}\left(e^{-x^2}\frac{dH_\alpha}{dx}\right) + 2\alpha e^{-x^2}H_\alpha = 0 \tag{3.10}$$

In each of the last three examples, just as in the case of the first two examples, there exist different sets of orthogonal functions depending on the boundary condtions imposed.