LECTURE 4

Series Solutions of the Heat Equation, Cont'd

1. Summary of Sturm-Liouville Theory

A Sturm-Liouville problem is a one parameter family of differential equations / boundary problems of the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda r(x))\, y = 0 \tag{4.1}$$

$$c_1 y(a) + c_2 y'(a) = 0 \tag{4.2}$$

$$d_1 y(b) + d_2 y'(b) = 0 \tag{4.3}$$

The relevant facts are

1. For all but a discrete set $\{\lambda_n \mid n \in \mathbb{N}\}$ of choices of λ, there are no solutions to (4.1) - (4.3). The λ_n are referred to as *eigenvalues* of the Sturm-Liouville problem, and corresponding solutions ϕ_{λ_n} are referred to as the *eigenfunctions* of the Sturm-Liouville problem.
2. If $\phi_{\lambda_n}(x)$ and $\phi_{\lambda_m}(x)$ are solutions to (4.1) - (refl04-0c) and $\lambda_n \neq \lambda_m$, then
$$\int_a^b \phi_{\lambda_n}(x)\phi_{\lambda_m}(x) r(x)dx = 0 \quad .$$
3. If the eigenvalues λ_n are nondegenerate, then the functions
$$\beta_m(x) = \frac{\phi_{\lambda_m}(x)}{\left|\int_a^b (\phi_{\lambda_m}(x))^2\, r(x)dx\right|^{1/2}}$$
form a complete orthonormal basis for space of quasi-smooth functions on the interval $[a, b]$. If some of the eigenvalues λ_n are degenerate, then the Gram-Schmid orthogonalization procedure can be used to construct an orthonormal basis from the eigenfunctions ϕ_{λ_n}.
4. Every quasi-smooth function $f : [a, b] \to \mathbb{R}$, can be expanded as
$$f(x) = \sum_{n=1}^{\infty} \alpha_n \beta_n(t)$$
where the coefficients α_n are determined by
$$\alpha_n = \int_a^b f(x)\beta_n(x) r(x)dx \quad .$$
This series converges in the sense that
$$\lim_{N\to\infty} \int_a^b \left(f(x) - \sum_{n=1}^{N} \alpha_n \beta_n(x)\right)^2 dx = 0 \quad .$$

2. Series Solutions Satisfying Homogeneous Boundary Conditions

We shall now attempt to construct a solution of the equations

$$\frac{\partial \phi}{\partial t} - a^2 \frac{\partial^2 \phi}{\partial x^2} = w(x,t) \tag{4.4}$$

$$\phi(x,0) = f(x) \tag{4.5}$$

$$\phi(0,t) = 0 \tag{4.6}$$

$$\phi(L,t) = 0 \tag{4.7}$$

in the region $0 \leq x \leq L$, $t > 0$. Note that this differs from the example in Lecture 1 because now the PDE is inhomogeneous (the example in Lecture 1 had $w(x,t) = 0$).

We shall begin by generalizing the ansatz we used in the separation of variables technique. Instead of demanding

$$\phi(x,t) = F(x)G(t)$$

we shall suppose

$$\phi(x,t) = \sum_{n \in \mathbb{N}} \alpha_n(t)\beta_n(x) \quad , \tag{4.8}$$

where $\{\beta_n, n \in \mathbb{N}\}$ is a complete basis for the set of twice differentiable functions on the interval $[0, L]$. This expansion is interpreted as follows: for any fixed time t, the solution $\phi(x,t)$ can be regarded as a function $\psi_t(x)$ of x alone. But "every" function of x on the interval can be represented as an infinite linear combination of the basis functions β_n; thus,

$$\psi_t(x) = \sum_{n \in \mathbb{N}} \alpha_{t,n}\beta_n(x) \quad .$$

Setting $\alpha_n(t) = \alpha_{t,n}$, we arrive at the expansion (4.2).

Now although any complete set $\{\beta_n\}$ will do, substantial simplifications will occur if the basis $\{\beta_n\}$ is chosen in a way that complements the original problem. More precisely, we shall choose the functions $\beta_n(x)$ to be the complete basis corresponding to the Sturm-Liouville problem

$$\beta'' + \lambda\beta = 0 \tag{4.9}$$

$$\beta(0) = 0 \tag{4.10}$$

$$\beta(L) = 0 \quad . \tag{4.11}$$

This particular choice will be justified by the relative ease by which we obtain a solution of (4.4) - (4.7). At present we remark only that when one tries to solve the homogeneous equation corresponding to (4.4) by separation of variables one is lead to the initial value problem

$$\begin{aligned} F''(x) - \frac{C}{a^2}F(x) &= 0 \\ F(0) &= 0 \\ F(L) &= 0 \end{aligned}$$

which is just the Sturm-Liouville problem (4.9) - (4.11).

As we discovered last time, the nontrivial solutions of (4.9) - (4.11) occur only when

$$\lambda = -\frac{n^2\pi^2}{L^2}$$

and

$$\beta_n(x) = \sin\left(\frac{n\pi}{L}x\right) \quad .$$

We thus set

$$\phi(x,t) = \sum_{n=1}^{\infty} \alpha_n(t) \sin\left(\frac{n\pi}{L}x\right) \quad . \tag{4.12}$$

Similarly, we set

$$\omega(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi}{L}x\right) \tag{4.13}$$

where the coefficients $w_n(t)$ are determined by

$$w_n(t) = \frac{2}{L}\int_0^L w(x,t) \sin\left(\frac{n\pi}{L}x\right) dx \quad .$$

Plugging (4.12) into the original differential equation (4.4) yields

$$\sum_{n=1}^{\infty} \left(\alpha_n'(t) - a^2\frac{n^2\pi^2}{L^2}\right) \sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi}{L}x\right),$$

If we multiply both sides of this equation by $\sin\left(\frac{m\pi x}{L}\right)$ and then integrate from 0 to L we obtain

$$\int_0^L \sum_{n=1}^{\infty} \left(\alpha_n'(t) - a^2\frac{n^2\pi^2}{L^2}\right) \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

Using the identities

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{L}{2}\delta_{mn}$$

we then obtain

$$\sum_{n=1}^{\infty} \left(\alpha_n'(t) - a^2\frac{n^2\pi^2}{L^2}\right) \frac{L}{2}\delta_{mn} = \sum_{n=1}^{\infty} w_n(t)\frac{L}{2}\delta_{mn}$$

or

$$\alpha_n'(t) - \frac{n^2a^2\pi^2}{L^2}\alpha_n(t) = w_n(t) \quad . \tag{4.14}$$

Equation (4.14) is a first order linear differential equation with constant coefficients; its general solution is

$$\alpha_n(t) = e^{-\frac{n^2\pi^2a^2}{L^2}t}\left[\int_0^t e^{\frac{n^2\pi^2a^2}{L^2}\tau} w_n(\tau)d\tau \quad + \quad c_n\right]$$

Thus,

$$\phi(x,t) = \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2a^2}{L^2}t}\left[\int_0^t e^{\frac{n^2\pi^2a^2}{L^2}\tau} w_n(\tau)d\tau \quad + \quad c_n\right] \sin\left(\frac{n\pi}{L}x\right) \quad . \tag{4.15}$$

Finally, to fix the constants c_n we impose the initial condition

$$\phi(x,0) = f(x) \quad .$$

This leads to

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$$

which, by applying again the orthogonality properties of the $\beta_n(x)$, leads to

$$c_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)dx \quad . \tag{4.16}$$

In summary, the solution to equations (4.4) - (4.7) is given by the following formula:

$$\phi(x,t) = \sum_{n=1}^{\infty} e^{-\frac{n^2\pi^2a^2}{L^2}t}\left[\int_0^t e^{\frac{n^2\pi^2a^2}{L^2}\tau} w_n(\tau)d\tau \quad + \quad \frac{2}{L}\int_0^L \sin\left(\frac{n\pi}{L}x\right) f(x)dx\right] \sin\left(\frac{n\pi}{L}x\right) \quad .$$

Homework Problems:

1. Prove (directly) that if ϕ_{λ_1} and ϕ_{λ_2} are solutions of

$$\begin{aligned} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda r(x))\, y &= 0 \\ y(a) &= 0 \\ y(b) &= 0 \end{aligned}$$

on the interval (a, b), respectively for $\lambda = \lambda_1$, and $\lambda = \lambda_2$, then

$$\int_a^b \phi_{\lambda_1}(x)\phi_{\lambda_2}(x)r(x)dx = 0 \quad .$$

if $\lambda_1 \neq \lambda_2$.

2. Discuss the implications of the Sturm-Liouville Theorem for the following ODE/BVP

$$\begin{aligned} \frac{d^2 f}{dx^2} + \lambda^2 f &= 0 \\ f(0) &= 0 \\ f(2\pi) &= 0 \end{aligned}$$

and their correspondence with the Fourier Theorem. (In other words, show that the Fourier theorem is a special case of the Sturm-Liouville theorem.)

3. (Problem 1.6.2 in the text.)

(a) For $0 < x < L$, solve the problem

$$\begin{aligned} \phi_t - a^2\phi_{xx} &= w(x,t) \\ \phi(0,t) &= 0 \\ \phi(L,t) &= 0 \\ \phi(x,0) &= f(x) \end{aligned}$$

by means of a series expansion involving the eigenfunctions of $\beta'' + \lambda\beta = 0$, $\beta(0) = 0$, $\beta(L) = 0$; where $w(x,t)$ and $f(x)$ are prescribed functions.

(b) If the end conditions are altered to read

$$\begin{aligned} \phi(0,t) &= 0 \\ \phi(L,t) + c\phi_x(L,t) &= 0 \end{aligned}$$

where $c > 0$ is a constant, find an appropriate set of eigenfunctions and obtain a series solution to the problem.