LECTURE 18

First Order Equations

EXAMPLE 18.1. Consider the following first order linear PDE

$$\phi_x + 2x\phi_y = y \tag{18.1}$$

subject to the boundary condition

$$\phi(0,y) = 1 + y^2 \qquad , \qquad \text{for} 1 < y < 2 \quad . \tag{18.2}$$

Suppose that $\phi(x,y)$ is a solution of this PDE/BVP. If we look along the curves of the form

$$y = f(x) \tag{18.3}$$

we have

$$\frac{d}{dx}\phi\left(x, f(x)\right) = \phi_x + f'(x)\phi_y \quad . \tag{18.4}$$

Thus, if we evaluate (18.1) along the curve

$$y = x^2 + C_1 \tag{18.5}$$

we get

$$\frac{d}{dx}\phi\left(x, x^2 + C_1\right) = \phi_x + 2x\phi_y = y = x^2 + C_1 \quad . \tag{18.6}$$

Integrating the extreme sides of (18.6) with respect to x we get

$$\phi\left(x, x^2 + C_1\right) = \frac{1}{3}x^3 + C_1 x + C_2 \qquad . \tag{18.7}$$

The constants C_1 and C_2 can be interpreted as follows. The curve (18.5) is a parabola that intersects the y-axis at the point C_1. From (18.7) it is clear that C_2 corresponds to the value of ϕ at the point $(0, C_1)$.

Equation (18.7) now tells us that by choosing a parabolic curve (by choosing C_1) and fixing the value C_2 of $\phi(x,y)$ at the point where this parabola intersects the y-axis, we can obtain the values of $\phi(x,y)$ at every other point along the parabolic curve.

From (18.2), we know the values of ϕ at all the points along the y-axis between $y = 1$ and $y = 2$. Using (18.7), we can thus compute the values of ϕ at all points in the region R of (x,y)-plane that is bounded by the curves $y = x^2 + 1$ and $y = x^2 + 2$.

To see this more explicitly, let $P = (x_1, y_1) \in R$. Setting

$$y_1 = x_1^2 + C_1 \tag{18.8}$$

we see that the parabolic curve through P intersects the y-axis at the point

$$C_1 = y_1 - x_1^2 \tag{18.9}$$

which we assume to lie between $y = 1$ and $y = 2$. According to (19.2), then the value of ϕ at the point $(0, y_o)$ is

$$C_2 = 1 + y_o^2 = 1 + \left(y_1 - x_1^2\right)^2 \quad . \tag{18.10}$$

Thus, equation (18.7) gives us

$$\phi\left(x_1, y_1\right) = x_1^2 + \left(y_1 - x_1^2\right) x_1 + \left(1 + \left(y_1 - x_1^2\right)^2\right) \quad . \tag{18.11}$$

This procedure has uniquely determined $\phi(x, y)$ in the region R, but has given us no information about ϕ outside the region R. In order to fix the functional form of ϕ outside of R, we could extend the boundary condition so that

$$\phi(0, y) = g(y) \qquad , \qquad \forall\, y \in \mathbb{R} \quad .$$

However, no matter how we extend the boundary conditions along the y-axis, as long as

$$g(y) = 1 + y^2$$

in the interval $1 < y < 2$, the functional form of $\phi(x, y)$ in the region R will not be affected.

Example 18.2. Consider now the following PDE/BVP:

$$\begin{aligned} x\phi_x + y\phi_y &= 1 + y^2 \\ \phi(x, 1) &= x + 1 \quad . \end{aligned} \tag{18.12}$$

In this example, we could divide through by x, to get

$$\phi_x + \frac{y}{x}\phi_y = \frac{1 + y^2}{x} \tag{18.13}$$

and then try to construct solutions along curves $y = f(x)$ with

$$f'(x) = \frac{y}{x} \quad . \tag{18.14}$$

However, such a formulation would introduce singularities at $x = 0$ which could be avoided.

So instead, consider a curve in the (x, y)-plane defined by some function $\gamma : \mathbb{R} \to \mathbb{R}^2$; $t \;\mapsto\; (x(t), y(t))$. If

$$\frac{dx}{dt} = x \qquad , \qquad \frac{dy}{dt} = y \quad , \tag{18.15}$$

then

$$\frac{d}{dt}\phi\left(\gamma(t)\right) = x\phi_x + y\phi_y \quad . \tag{18.16}$$

Thus, along the curve γ, the PDE in (18.12) can be written

$$\frac{d}{dt}\phi\left(\gamma(t)\right) = 1 + \left(y(t)\right)^2 .$$

Solving the differential equations (18.15) for $x(t)$ and $y(t)$, we can make this equation for $\phi\left(\gamma(t)\right)$ even more explicit:

$$\begin{aligned} \frac{dx}{dt} &= x \qquad \Rightarrow \qquad x = C_1 e^t \\ \frac{dy}{dt} &= y \qquad \Rightarrow \qquad y = C_2 e^t \end{aligned}$$

so

$$\frac{d}{dt}\left(\phi \circ \gamma\right) = 1 + \left(C_2 e^t\right)^2 \quad . \tag{18.17}$$

Integrating (18.17) produces

$$\phi\left(C_1 e^t, C_2 e^t\right) = t + \frac{1}{2} C_2 e^{2t} + C_3 \quad . \tag{18.18}$$

Without loss of generality, we can assume that the curve γ crosses the line $y = 1$ when $t = 0$. Then if the curve γ crosses the line $y = 1$ at the point x_o, we would have

$$C_1 = x_o \qquad , \qquad C_2 = 1 \quad . \tag{18.19}$$

Now consider an arbitrary point $P = (x_1, y_1)$ in the first quadrant. Suppose γ passes through P, then

$$\begin{aligned} x &= x_o e^t \\ y &= e^t \end{aligned} \tag{18.20}$$

for some t. Solving (18.20) for x_o and t we get

$$\begin{aligned} t &= \ln|y| \\ x_o &= \tfrac{x}{y} \end{aligned} \quad . \tag{18.21}$$

We are also given the boundary condition

$$\phi(x_o, 1) = x_o + 1 \quad . \tag{18.22}$$

Evaluating (18.18) at $t = 0$ yields

$$x_o + 1 = 0 + \frac{1}{2} + C_3 \quad ,$$

hence

$$C_3 = x_o + \frac{1}{2} \quad . \tag{18.23}$$

Equations (18.19), (18.21), and (18.23), now enable us to rewrite (18.18) as

$$\begin{aligned} \phi(x,y) &= t + \tfrac{1}{2} C_2 e^{2t} + C_3 \\ &= \ln|y| + \tfrac{1}{2}\left(e^{2\ln|y|}\right) + \left(x_o + \tfrac{1}{2}\right) \\ &= \ln|y| + \tfrac{1}{2} y^2 + \tfrac{x}{y} + \tfrac{1}{2} \end{aligned} \quad . \tag{18.24}$$

We conclude that the solution of (18.12) is

$$\boxed{\phi(x,y) = \ln|y| + \frac{1}{2} y^2 + \frac{x}{y} + \frac{1}{2}}$$