APPENDIX C

Solutions to Problem Set 3

1.

Prove (directly) that if ϕ_{λ_1} and ϕ_{λ_2} are solutions of

$$\begin{aligned}\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (q(x) + \lambda r(x))\, y &= 0 \\ y(a) &= 0 \\ y(b) &= 0\end{aligned}$$

on the interval (a,b), respectively for $\lambda = \lambda_1$, and $\lambda = \lambda_2$, then

$$\int_a^b \phi_{\lambda_1}(x)\phi_{\lambda_2}(x)r(x)dx = 0 \quad .$$

if $\lambda_1 \neq \lambda_2$.

(See Lecture 3) □

2.

Discuss the implications of the Sturm-Liouville Theorem for the following ODE/BVP

$$\begin{aligned}\frac{d^2 f}{dx^2} + \lambda^2 f &= 0 \\ f'(0) &= 0 \\ f'(\pi) &= 0\end{aligned} \tag{C.1}$$

and their correspondence with the Fourier Theorem. (In other words, show that the Fourier theorem is a special case of the Sturm-Liouville theorem.)

The general solution of

$$f'' + \lambda^2 f = 0$$

is given by

$$f(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x).$$

The boundary condition $f'(0) = 0$ implies

$$0 = -c\lambda \sin(0) + c_2 \lambda \cos(0) = c_2$$

so $c_2 = 0$. The boundary condition $f'(L) = 0$ then implies

$$0 = -c_1 \lambda \sin(\lambda L).$$

which will be satisfied (for non-trivial c_1) if and only if

$$\lambda = \frac{n\pi}{L} \ , \ n = 0, 1, 2, 3,$$

The Sturm-Louisville Theorem then tells us that the solutions

$$\beta_n(x) = \left[\int_0^L \cos^2\left(\frac{n\pi}{L}x\right) dx\right]^{-1/2} \cos\left(\frac{n\pi}{L}x\right) = \begin{cases} \sqrt{L} & \text{if} \quad n = 0 \\ \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi}{L}x\right) & \text{if} \quad n = 1, 2, 3, \ldots \end{cases}$$

will form a complete orthonormal set of basis functions for the interval $[0, L]$. More explicitly, we have

$$(\beta_n, \beta_m) = \int_0^L \beta_n(x)\beta_m(x)dx = \frac{2}{L}\int \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) dx = \delta_{mn}$$

and any continuous function f on the interval $[0, L]$ can be approximated by a series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} \alpha_n \beta_n(x) \tag{C.2}$$

where

$$\alpha_n = \sqrt{\frac{2}{L}}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right) dx.$$

If we set

$$a_n = \sqrt{\frac{L}{2}}\alpha_n = \int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right) dx$$

then we can rewrite (C.2) as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

with

$$a_n = \int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right) dx$$

which is just the usual Fourier Cosine Series expansion of $f(x)$. □

3. (Problem 1.6.2 in the text)

(a) For $0 < x < L$, solve the problem

$$\begin{aligned} \phi_t - a^2\phi_{xx} &= w(x,t) \\ \phi(0,t) &= 0 \\ \phi(L,t) &= 0 \\ \phi(x,0) &= f(x) \end{aligned}$$

by means of a series expansion involving the eigenfunctions of $\beta'' + \lambda\beta = 0$, $\beta(0) = 0$, $\beta(L) = 0$; where $w(x,t)$ and $f(x)$ are prescribed functions.

(See Lecture 4.)

(b) If the end conditions are altered to read

$$\begin{aligned} \phi(0,t) &= 0 \\ \phi(L,t) + c\phi_x(L,t) &= 0 \end{aligned} \tag{C.3}$$

where $c > 0$ is a constant, find an appropriate set of eigenfunctions and obtain a series solution to the problem.

Applying separation of variables to the homogeneous version of the PDE we arrive at the following pair of coupled ODEs:

$$\begin{aligned} T' + \Lambda T &= 0 \\ X'' + \frac{\Lambda}{a^2} X &= 0 \end{aligned} \tag{C.4}$$

Note first that only the second equation will serve as the ODE of a Sturm-Liouville problem (it is the only second order linear equation). Secondly, note that by setting

$$\begin{aligned} X(0) &= 0 \\ X(L) + cX'(L) &= 0 \end{aligned} \tag{C.5}$$

we can assure that the first two boundary conditions are satisfied. We thus are lead to consider the following Sturm-Liouville problem

$$\begin{aligned} \beta'' + \lambda^2 \beta &= 0 \\ \beta(0) &= 0 \\ \beta(L) + c\beta'(L) &= 0 \end{aligned} \tag{C.6}$$

Now the general solution of the ODE for this Sturm-Liouville problem is

$$\beta(x) = A\cos(\lambda x) + B\sin(\lambda x) \tag{C.7}$$

in order to satisfy the first boundary condition $\beta(0) = 0$ we must set $A = 0$. Let us now impose the second boundary condition

$$0 = B\sin(\lambda L) + cB\cos(\lambda L)\,. \tag{C.8}$$

Then the second boundary condition now requires

$$0 \quad = \quad \sin(\lambda L) + c\lambda\cos(\lambda L) \tag{C.9}$$

or

$$-c\lambda = \tan(\lambda L) \quad . \tag{C.10}$$

This is unfortunately a transcendental equation for λ. It does, however, have an infinite (yet countable) number of roots. To see this, we note that the function $\tan(Lx)$ is periodic with period $\frac{\pi}{L}$, and within any interval $I_n = \left(\frac{1}{L}\left(n\pi - \frac{\pi}{2}\right), \frac{1}{L}\left(n\pi + \frac{\pi}{2}\right)\right)$ it is monotonically increasing and maps I_n onto the real line. Therefore, the graph of $\tan(Lx)$ intersects the line $y = -cx$ once and only once in each interval I_n. We can now apply Newton's method to write down an algorithm for finding a root of

$$f(\lambda) = \tan(L\lambda) + c\lambda = 0 \tag{C.11}$$

in each interval I_n. More explicitly, if we set

$$r_{n,1} = \frac{n\pi}{L} \in I_n \tag{C.12}$$

and then define $r_{n,2}$, $r_{n,3}$, ... recursively by the formula

$$r_{n,i+1} = r_{n,i} - \frac{f(r_{n,i})}{f'(r_{n,i})} = \frac{\tan(Lr_{n,i}) + cr_{n,i}}{L\sec^2(r_{n,i}) + c} \quad , \quad i = 2, 3, 4, \ldots \tag{C.13}$$

then

$$\lambda_n = \lim_{i\to\infty} r_{n,i} \tag{C.14}$$

will be the root of (C.10).

Let us assume that this has now been carried out - so have obtained an infinite set of solutions λ_n of (C.11). Note that both sides of (C.10) are odd functions of λ. Therefore, if λ is a solution so is $-\lambda$. Note also that the S-L functions $\sin(\lambda x)$ and $\sin(-\lambda x) = -\sin(\lambda x)$ are not linearly independent. For this reason, we can neglect the negative roots of (C.10). We can also neglect the trivial root $\lambda = 0$ since it corresponds to the trivial function $\sin(0x) = 0$.

In summary, let $r_{n,1} = \frac{n\pi}{L}$, $n = 1, 2, 3, \ldots$, and let us define numbers λ_n by (C.12), (C.13) and (C.14). Then the functions

$$\beta_n(x) = \sin(\lambda_n x)$$

will constitute a complete set of eigenfunctions for the interval $(0, L)$ corresponding to the solutions of the Sturm-Liouville problem (C.6). Moreover, if we renormalize the β_n by setting

$$\gamma_n(x) = \frac{\sin(\lambda_n x)}{\left|\int_0^L \sin^2(\lambda_n x)\, dx\right|^{1/2}} \tag{C.15}$$

then the set $\{\gamma_n \mid n \in \mathbb{N}\}$ will constitute a complete **orthonormal** basis for the space of continuous functions on the interval $(0, L)$: i.e.,

$$\int_0^L \gamma_n(x)\gamma_m(x)dx = \begin{cases} 1 \text{ if } n = m \\ 0 \text{ if } n \neq m \end{cases} \quad . \tag{C.16}$$

We can now apply the completeness property of the eigenfuctions γ_n to write

$$\phi(x,t) = \sum_{n=1}^{\infty} a_n(t)\gamma_n(x) \quad . \tag{C.17}$$

Because, by construction,

$$\begin{aligned} \gamma(0) &= 0 \\ \gamma(L) + c\gamma'(L) &= 0 \end{aligned}$$

the boundary conditions

$$\begin{aligned} \phi(0,t) &= 0 \\ \phi(L,T) + c\phi_x(L,t) &= 0 \end{aligned}$$

are automatically satisfied by this ansatz. Plugging (C.17) into the original PDE produces

$$\sum_{n=1}^{\infty} a_n'(t)\gamma_n(x) - \sum_{n=1}^{\infty} a^2 a_n(t)\gamma_n''(x) = w(x,t) \tag{C.18}$$

Applying the completeness property of the γ_n we can replace the right hand side of (C.18) by

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\gamma_n(x) \tag{C.19}$$

where the coefficients $w_n(t)$ are determined by

$$w_n(t) = \int_0^L \gamma_n(x) w(x,t)\, dx \quad . \tag{C.20}$$

Thus, (C.18) is equivalent to

$$\begin{aligned} 0 &= \textstyle\sum_{n=1}^{\infty} a_n'(t)\gamma_n(x) - \sum_{n=1}^{\infty} a^2 a_n(t)\gamma_n''(x) - \sum_{n=1}^{\infty} w_n(t)\gamma_n(x) \\ &= \textstyle\sum_{n=1}^{\infty} \left(a_n'(t)\gamma_n(x) - a^2 a_n(t)\gamma_n''(x) - w_n(t)\gamma_n(x)\right) \\ &= \textstyle\sum_{n=1}^{\infty} \left(a_n'(t)\gamma_n(x) + a^2\lambda_n^2 a_n(t)\gamma_n(x) - w_n(t)\gamma_n(x)\right) \\ &= \textstyle\sum_{n=1}^{\infty} \left(a_n'(t) + a^2\lambda_n^2 a_n(t) - w_n(t)\right)\gamma_n(x) \end{aligned} \tag{C.21}$$

In the second step we have simply used the fact that the γ_n (like their un-normalized predecessors β_n) are by definition solutions of

$$\gamma_n'' + \lambda_n^2\gamma_n = 0 \quad .$$

Multiplying the extreme sides of (C.21) by $\gamma_m(x)$ and integrating between 0 and L yields

$$\begin{aligned} 0 &= \int_0^L \sum_{n=1}^{\infty} \left(a_n'(t) + a^2\lambda_n^2 a_n(t) - w_n(t)\right)\gamma_n(x)\gamma_m(x)\,dx \\ &= \sum_{n=1}^{\infty} \left(a_n'(t) + a^2\lambda_n^2 a_n(t) - w_n(t)\right)\int_0^L \gamma_n(x)\gamma_m(x)\,dx \\ &= \sum_{n=1}^{\infty} \left(a_n'(t) + a^2\lambda_n^2 a_n(t) - w_n(t)\right)\delta_{nm} \\ &= a_m'(t) + a^2\lambda_m^2 a_m(t) - w_m(t) \end{aligned}$$

Thus, in order (C.17) to satisfy the PDE in (C.3) the coefficient functions $a_n(t)$ must thus be solutions of

(C.22) $$a_n'(t) + a^2\lambda_n^2 a_n(t) = w_n(t) \quad .$$

This is first order linear ODE for which the general solution is well known (see e.g., Boyce and DiPrima, Chapter 2); it is given by the formula

(C.23) $$a_n(t) = e^{-a^2\lambda_n^2 t}\left(\int_0^t e^{a^2\lambda_n^2 s} w_n(s)\,ds + c_n\right)$$

with c_n a constant representing the value of $a_n(t)$ when $t = 0$.

To fix the constants c_n we now impose the last boundary condition

(C.24) $$f(x) = \phi(x,0) = \sum_{n=1}^{\infty} a_n(0)\gamma_n(x) = \sum_{n=1}^{\infty} c_n\gamma_n(x) \quad .$$

Multiplying the extreme sides of this equation by $\gamma_m(x)$, integrating both sides between 0 and L, and employing the orthonormality properties (C.16) of the $\gamma_n(x)$ we obtain

(C.25) $$c_n = \int_0^L f(x)\gamma_n(x)\,dx \quad .$$

In summary, the solution to

$$\begin{aligned} \phi_t - a^2\phi_{xx} &= w(x,t) \\ \phi(0,t) &= 0 \\ \phi(L,t) + c\phi_x(L,t) &= 0 \\ \phi(x,0) &= f(x) \end{aligned}$$

is given by

$$\phi(x,t) = \sum_{n=1}^{\infty} e^{-a^2\lambda_n^2 t}\left(\int_0^t e^{a^2\lambda_n^2 s}\left(\int_0^L \gamma_n(x')w(x',s)\,dx\right)ds + \int_0^L f(x')\gamma_n(x')\,dx'\right)\gamma_n(x)$$

where the constants λ_n are determined by (C.12), (C.13), and (12); and the functions $\gamma_n(x)$ are defined by (C.15). □

4. (Problem 1.8.1 in text)

Use a series expansion technique to solve the problem

(C.26) $$\begin{aligned} \frac{\partial\phi}{\partial t} - a^2\frac{\partial^2\phi}{\partial x^2} &= 1 \\ \phi(x,0) &= 0 \\ \phi(0,t) &= t \\ \frac{\partial\phi}{\partial x}(L,t) + c\phi(L,t) &= 0 \quad , \end{aligned}$$

where $c > 0$ is a constant, in the region $t > 0$, $0 < x < L$.

Let's first convert this to a homogeneous problem. Set

(C.27) $$\psi(x,t) = \phi(x,t) + \zeta(x,t) \quad .$$

We want to choose $\zeta(x,t)$ so that

(C.28) $$\begin{aligned} \psi(x,0) &= 0 \\ \psi(0,t) &= 0 \\ \frac{\partial\psi}{\partial x}(L,t) + c\psi(L,t) &= 0 \end{aligned}$$

These equations lead to the following conditions on $\zeta(x,t)$

(C.29) $$\begin{aligned} \zeta(x,0) &= 0 \\ \zeta(0,t) &= -t \\ \zeta_x(L,t) + c\zeta(L,t) &= 0 \quad . \end{aligned}$$

The first two conditions will be satisfied by any function $\zeta(x,t)$ of the form

(C.30) $$\zeta(x,t) = f(x)t - t$$

with $f(0) = 0$. The second condition puts a restriction on the choice of f; viz.,

$$f'(L)t + c\,(f(L)t - t) = 0,$$

or

$$f'(L) + cf(L) = c \quad .$$

To find a solution of this equation we set

$$f'(x) = c - f(L)c$$

and integrate both sides. The result corresponding to the initial condition $f(0) = 0$ is

$$f(x) = cx - f(L)cx \quad .$$

Plugging in $x = L$ and solving for $f(L)$ we get

$$f(L) = \frac{cL}{1 + cL} \quad .$$

Thus, we can take

$$f(x) = cx - \frac{cL}{1+cL}cx = \frac{cx}{1+cL}$$

and

(C.31) $$\zeta(x,t) = t\left(\frac{cx}{1+cL} - 1\right) = t\left(\frac{c(x-L)-1}{1+cL}\right).$$

Applying the differential operator $\partial_t - a^2\partial_x^2$ to $\psi\ (x,t) = \phi(x,t) + \zeta(x,t)$ and using (C.26), we find

(C.32) $$\begin{aligned} \frac{\partial\psi}{\partial t} - a^2\frac{\partial^2\psi}{\partial x^2} &= \frac{cx}{1+cL} \\ \psi(x,0) &= 0 \\ \psi(0,t) &= 0 \\ \frac{\partial\psi}{\partial x}(L,t) + c\psi(L,t) &= 0 \quad . \end{aligned}$$

The boundary conditions for ψ are thus homogeneous.

To solve (C.32) we make the ansatz

$$\psi(x,t) = \sum_n \alpha_n(t)\beta_n(x) \quad ,$$

the $\beta_n(x)$ being a complete basis of functions coming from the Sturm-Louiville problem

$$\begin{aligned} y'' + \lambda^2 y &= 0 \\ y(0) &= 0 \\ y'(L) + cy(L) &= 0 \end{aligned}$$

(the differential equation with respect to the variable x coming from separation of variables of the homogeneous problem corresponding to (7)). The general solution of

$$y'' + \lambda^2 y = 0$$

satisfying $y(0) = 0$ is

$$y(x) = A\sin(\lambda x) \quad .$$

Imposing the boundary condition $y'(L) + cy(L) = 0$, requires

$$\lambda\cos(\lambda L) + c\sin(\lambda L) = 0$$

or

$$\tan(\lambda L) = -\frac{\lambda}{c} \tag{C.33}$$

Equation (C.33) has an infinite number of roots. Let $\{\lambda_n\}$ denote the set of consecutive positive roots of (C.33). The corresponding eigenfunctions are

$$y_n(x) = \sin(\lambda_n x) \quad .$$

(Since both sides of (C.33) are odd functions of λ, if λ_n is a root so is $-\lambda_n$; but the corresponding eigenfunctions are the same, except for a factor of -1. The reason why we consider only the positive roots of (8) is to remove this redundancy.) By Sturm-Louiville theory, these functions are all othogonal with respect to the inner product

$$(f,g) = \int_0^L f(x)g(x)dx \quad .$$

An explicit computation reveals

$$\begin{aligned} (y_n, y_m) &= \delta_{m,n}\left(-\frac{1}{2\lambda_n}\cos(\lambda_n x)\sin(\lambda_n x) + \frac{1}{2}x\right)\Big|_0^L \\ &= \delta_{m,n}\left(\frac{L}{2} - \frac{1}{2\lambda_n}\cos(\lambda_n L)\sin(\lambda_n L)\right) \\ &= \delta_{m,n}\left(\frac{L}{2} - \frac{1}{4\lambda_n}\sin(2\lambda_n L)\right) \end{aligned}$$

and so the functions

$$\beta_n(x) = \frac{1}{\left(\frac{L}{2} - \frac{1}{4\lambda_n}\sin(2\lambda_n L)\right)^{1/2}}\sin(\lambda_n x)$$

will form an orthonormal basis for the set of continuous functions on $(0, L)$. Now set

$$\psi(x,t) = \sum_n \alpha_n(t)\beta_n(x)$$

and write

$$\frac{cx}{1+cL} = \sum_n w_n\beta_n(x) \quad , \tag{C.34}$$

the γ_n being determined by

$$w_n = \frac{1}{\left(\frac{L}{2} - \frac{1}{4\lambda_n}\sin(2\lambda_n L)\right)^{1/2}}\int_0^L \left(\frac{cx}{1+cL}\right)\sin(\lambda_n x)\,dx \quad . \tag{C.35}$$

Plugging (C.34) and (C.35) into (C.32) yields

$$\sum_n \alpha_n'(t)\beta_n(x) + \sum_n a^2\lambda_n^2\alpha(t)\beta_n(x) = \sum_n w_n\beta_n(x)$$

or

(C.36)
$$\alpha_n'(t) + a^2\lambda_n^2\alpha(t) = w_n \quad .$$

The boundary condition $\psi(x,0) = 0$ implies $\alpha_n(t) = 0$. Recalling that the general solution solution to a first order, linear, nonhomogeneous, ordinary differential equation

$$y' + p(t)y = g(t)$$

with initial condition

$$y(0) = y_o$$

is

$$y(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(t)g(s)ds + y_o\right]$$

where

$$\mu(t) = \exp\left[\int_0^t p(x)dx\right]$$

(see Boyce and DiPrima, Sec. 2.2); we find that the solution of (11) satisfying $\alpha_n(0) = 0$ is

$$\alpha_n(t) = \frac{w_n}{a^2\lambda_n^2}\left(1 - e^{-a^2\lambda_n^2 t}\right) \quad .$$

Thus,

$$\psi(x,t) = \sum_n \frac{w_n}{a^2\lambda_{n^2}}\left(1 - e^{-a^2\lambda_n^2 t}\right)\beta_n(x) \qquad ,$$

Finally,

$$\begin{aligned}\phi(x,t) &= \psi(x,t) - \zeta(x,t) \\ &= \sum_n \frac{w_n}{a^2\lambda_{n^2}}\left(1 - e^{-a^2\lambda_n^2 t}\right)\beta_n(x) - t\left(\frac{c(x-L)-1}{1+cL}\right)\end{aligned}$$

with λ_n and w_n determined, respectively, by (C.33) and (C.35). □