A Fine Partitioning of Cells

Atlas of Lie Groups Workshop

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1. The setting in Atlas

Let me begin by talking a bit about the organization of the set of irreducible admissible representations from the atlas point of view.

1.1. G.

- G be a algebraic group defined over $\mathbb{C}$,
- $G^\vee$ its dual group,
- τ an outer automorphism determining an inner class of real forms
- $\delta \in Aut\,(G)$ a strong real form of G in the inner class of τ
- $G^\Gamma = G \rtimes \Gamma = G \rtimes \{1, \sigma\}$ with σ acting on G by δ

1.2. **Irreducible admissible representations.**

From the Atlas point of view, which is essentially a derivative of the Langlands point of view after several reductions/translations (Vogan-Zuckerman-Knapp-Adams-duCloux) the irreducible (admissible) representations (with integral infinitesimal character) are parameterized by triples (x, y, λ) where

- $x \in G^\Gamma - G$, $x^2 \in Z\,(G)$, a representative of a strong real form of G
- $y \in (G^\vee)^\Gamma - G^\vee$, $y^2 \in Z\,(G^\vee)$, a representative of a strong real form of $G^\vee$
- $\lambda \in {}^d\mathfrak{t} \cong \mathfrak{t}^*$

Remark 1.1. In much of the atlas documentation you will also see x and y regarded as representing a pair $(\mathcal{O}, \mathcal{O}^\vee)$, were $\mathcal{O}$ and $\mathcal{O}^\vee$ are, respectively, a K-orbit in G/B and a $K^\vee$-orbit in $G^\vee/B^\vee$. This is an equivalent parameterization -

1.3. **Blocks of Representations.**

By restricting the parameters x and y, respectively, to correspond to particular (equivalence classes of) strong real forms, the set $\widehat{G}_{adm}\,(\lambda)$ of irreducible admissible $(\mathfrak{g}, K)$-modules with integral infinitesimal character λ can split into "blocks" of representations. From a purely atlas-centric point of view the notion of blocks is useful as it provides a minimal partitioning of $\widehat{G}_{adm}\,(\lambda)$ into subsets for which KL computations are self-contained. But it turns out that blocks of representations correspond to the collecting together of all irreducible admissible representations into groups connected by a non-trivial Ext.

Below is the beginning of the atlas session in which the KL polynomials for the "big block" of split E8 is computed; we remark that the choice of a weak real form at the user interface level is actually implemented as a choice of a strong real from within the software itself.

```
real:  type
Lie type:  E8
enter inner class(es):  split
main:  klwrite
(weak) real forms are:
0:  e8
```

```
1:  e8(e7.su(2))
2:  e8(R)
enter your choice:  2
possible (weak) dual real forms are:
0:  e8
1:  e8(e7.su(2))
2:  e8(R)
enter your choice:  2
```

1.4. **Cells of Representations.** Within a block we can collect together those representations which belong to the same cell. Recall from Peter's talk that a cell is an equivalence class of representations where

$$X \sim Y \quad \Longleftrightarrow \quad \begin{array}{l} Y \text{ is a subquotient of } X \otimes F \text{ for some f.d.r. } F \text{ and} \\ X \text{ is a subquotient of } Y \otimes F' \text{ for some f.d.r. } F' \end{array}$$

and that the representations in the cell share the same associated variety. In particular, all the representations in a cell share the same Gelfand-Kirillov dimension.

The cell decomposition of a block is obtainable by the `wcells`, `wgraph`, or `extractgraph` commands of Atlas. Below is beginning of an atlas session in which the wcells command in run on the big block of F4

```
empty:  type
Lie type:  F4
enter inner class(es):  s
main:  wcells
(weak) real forms are:
0:  f4
1:  f4(so(9))
2:  f4(R)
enter your choice:  2
possible (weak) dual real forms are:
0:  f4
1:  f4(so(9))
2:  f4(R)
enter your choice:  2
entering block construction ...
228
done
computing kazhdan-lusztig polynomials ...
335
done

Name an output file (return for stdout, ?  to abandon):
// cell #0
0:{}:{}
// cell #1
0:{1}:{(3,1)}
1:{3}:{(2,1),(3,1)}
2:{4}:{(1,1)}
3:{2}:{(0,1),(1,1),(4,1)}
4:{3}:{(3,1),(5,1)}
5:{4}:{(4,1)}
// cell #2
0:{3,4}:{(3,1),(6,1)}
```

```
1:{1,4}:{(2,1),(3,1)}
2:{1,3}:{(1,1),(4,1),(5,1)}
3:{2,4}:{(0,1),(1,1),(4,1),(5,1)}
4:{3}:{(3,1)}
5:{2}:{(2,1),(6,1)}
6:{3}:{(5,1),(7,1)}
7:{4}:{(6,1)}
..
```

1.5. **Tau-invariants.** The output of the `wcells` command contains a lot of information about the representations in a block. The elements of a cell are labeled by (strictly internal) indices from 0 to $(m-1)$ where m is the number of elements in the cell; rather than pairs (x, y).[1] Following a cell index number i is a set $\{t_{i,1}, \ldots, t_{i,\ell_i}\}$ which is the "tau-invariant" of the cell element. This is actually an invariant of the primitive ideal corresponding to the annihilator of the irreducible $(\mathfrak{g}, K)$-module corresponding to $i \sim (x, y, \lambda)$. The tau-invariant is a set of indices of simple roots that, roughly speaking, prescribes the "direction" in which the correponding primitive ideal sits relative to the minimal primitive of ideal of infinitesimal character λ.

A little more precisely. Let $Prim_\rho$ be the set of primitive ideals of infinitesimal character ρ endowed with the natural partial ordering by inclusion.

$$I \leq I' \quad \Longleftrightarrow \quad I \subseteq I'$$

Then there is a unique maximal primitive ideal within $Prim_\rho$ (the annihilator of the trivial representation) and a unique minimal primitive ideal I_0 which is the annihilator of the irreducible Verma module $M_{-\rho}$ of highest weight -2ρ. The other primitive ideals of infinitesimal character ρ can be thought of as sitting on the vertices of a Hasse diagram associated with the above partial ordering. Since the minimal primitive ideal is contained in every primitive ideal, every primitive ideal in the $Prim_\rho$ is connected to I_0 by certain sequences of inclusions $I \supset I' \supset \cdots \supset I^{(k)} \supset I_0$, which can be visualized as certain directed paths through a Hasse diagram. It turns out that the penultimate primitive ideals in such a sequence are always primitive ideals of the form $Ann\,(M_{-s\rho}/M_{-\rho})$ where $M_{-s\rho}$ is the Verma module of highest weight $-s\rho - \rho$, s being a reflection by a simple root in W. For a given primitive ideal I let $\tau\,(I)$ denote the set of simple roots s for which $Ann\,(M_{-s\rho}/M_{-\rho})$ sits between I and I_0 in the Hasse diagram of $Prim_\rho$. In other words, the tau invariant $\tau\,(I)$ is the set of next-to-last-stops on the paths from I to I_0.

I should perhaps remark that the sets $\{t_{i,1}, \ldots, t_{i,\ell_i}\}$ that occur in the output of `wcells` are actually the indices of the simple roots that lie in the descent set of the representation indexed by i. The identification with tau-invariants comes via

Theorem 1.2. *If X is an irreducible $(\mathfrak{g}, K)$-module, then the tau-invariant of $Ann\,(X)$ is equal to the descent set of X.*

Note that in the example above, some cell elements share the same tau-invariant. This, however, does not mean that they share same annihilator; it simply means that their annihilators lie in the same direction(s) from I_0. On the other hand, since $\tau\,(X) \equiv \tau\,(Ann\,(X))$,

Fact 1.3. *If X, Y are two irreducible $(\mathfrak{g}, K)$-modules such that $Ann\,(X) = Ann\,(Y)$, then $\tau\,(X) = \tau\,(Y)$*

1.6. **Edges.** The output of the wcells command contains one last bit of cell element data; the *edges* and *mulitiplicities* of the Wgraph "star"[2] originating from a cell element. If (j, m) is listed in the edge/multiplicity data of a cell element i then the representation π_j corresponding to (cell index) j occurs in the HC module $\pi_i \otimes \mathfrak{g}$ with multiplicity m (π_i being the irreducible HC module corresponding to cell index i).

[1] The exact correspondence between the internal cell indices i and the pairs (x, y) of real forms can be deduced from the output of the `extractgraph` and `block` commands.

[2] Here I mean the Wgraph of the cell which is the restriction of the Wgraph of the block to the cell.

1.7. **Primitive Ideals.** In the above we allude to the possibility of grouping together cell elements which share the same primitive ideal; however, Atlas will not do that for us. Yet, ...

2. A fine partioning of cells

What one can obtain from immediately from the output of the `wcells` command is a partitioning of a cell into subcells with the same tau-invariant. As remarked above, this is compatible with the partitioning of a cell via primitive ideals, but it is much coarser. However, besides having the same tau-invariant, representations with the same primitive ideal also have the property that their collections of tau-invariants of their edge vertices are the same. By this I mean the following. Let $\tau_0(i) = \{t_{i,1}, \ldots, t_{i,m}\}$ denote the tau invariant of vertex i of a cell. Let $\mathbf{e}(i) = \{e_{i,1}, \ldots, e_{i,k}\}$ be the set of edge vertices for vertex i, and let

$$\tau_1(i) = \{\tau_0(e_{i,1}), \ldots, \tau_0(e_{i,k})\}$$

be the corresponding set (without multiplicity) of tau-invariants of the edge vertices of vertex i. If two vertices i, j share the same primitive ideal then

$$\tau_1(i) = \tau_1(j)$$

This equivalence relation still fails to complete separate the representations in a cell into subgroups with a common primitive ideal; however, it **is** compatible with the partitioning by primitive ideals and it **is** compatible with the partitioning than by tau-invariants.

So what to do next? Simply repeat. That is, set

$$\tau_2(i) = \{\tau_1(e_{i,1}), \ldots, \tau_1(e_{i,k})\}$$

and group together vertices with the same τ_0, the same τ_1 and the same τ_2. If you continue this process until the sub-partitioning process stabilizes (as it must since the cells are finite), and do this for, say, all real forms of all exceptional groups, then the following empirical fact emerges:

Fact 2.1. *The number of elements in the infinite order (stable) partitioning of a cell is always the dimension of a special representation of the Weyl group of G.*

What makes this so striking is the following.

Fact 2.2. *Attached (by other means [Lu]) to each cell C is a unique special representation σ_C of W, and*

$$\#\{Ann(X) \mid X \text{ an irreducible HC module in } C\} = \dim \sigma_C$$

That is, the number of distinct primitive ideals arising from a given cell is equal to the dimension of the special representation of W attached to that cell.

Conjecture 2.3. *The infinite ordering partitioning of a cell corresponds to a partitioning of the cell by primitive ideals.*

Patrick Polo asked the following question. Why should it be that representations with the same primitive ideal share the same τ_1 invariant (as well as the higher derived τ-invariants, $\tau_2, \ldots$)? David offered the following explanation.

Fix a irreducible $(\mathfrak{g}, K)$-module X of infinitesimal character ρ, and consider $I = Ann(X)$. $X \otimes \mathfrak{g}$ is a $(\mathfrak{g}, K)$-module that is also a faithful module for $U(\mathfrak{g})/I \otimes U(\mathfrak{g})/Ann(\mathfrak{g})$. You can diagonally embed $U(\mathfrak{g})$

$$\Delta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})/I \otimes U(\mathfrak{g})/Ann(\mathfrak{g})$$

Then

$$\Delta^{-1}(U(\mathfrak{g})/I \otimes U(\mathfrak{g})/Ann(\mathfrak{g})) = Ann_{U(\mathfrak{g})}(X \otimes \mathfrak{g})$$

depends only on I. Now $X \otimes \mathfrak{g}$ will decompose as

$$\begin{aligned} X \otimes \mathfrak{g} = &\ \text{several copies of } X \\ &+ \bigoplus Y_i \text{ (irreducible } (\mathfrak{g}, K)\text{-module in same cell as } X\text{; in fact, Wgraph neighbors of } X) \\ &+ (\mathfrak{g}, K)\text{-modules of lower GK-dim (bigger annihilators)} \end{aligned}$$

Therefore, the set of minimal primes in $Ann\,(X \otimes \mathfrak{g})$ will consist of $\{I, Ann\,(Y_i)\}$. Since $Ann\,(X \otimes \mathfrak{g})$ depends only on the primitive ideal I containing X, the set of minimal primes in $Ann\,(X \otimes \mathfrak{g})$, will also depend only on I and so if we have two $(\mathfrak{g}, K)$-modules X, X' in the same cell and consider the sets of primitive ideals corresponding to the Wgraph neighbors of X, X' we must have

$$\{Ann\,(Y_1)\,, \ldots, Ann\,(Y_k)\} \quad = \quad \left\{Ann\,(Y_1')\,, \ldots, Ann\left(Y_\ell'\right)\right\}$$

This implies

$$\tau_1\,(X) = \tau_1\,(X')$$

and, in fact, infers the equality of the higher derived tau-invariants as well.

References

[Lu] G. Lusztig, *A class of irreducible representations of a Weyl group* Nederl. Akad. Wetensch. Indag. Math. **41** (1979), no. 3, 323–335.

[M] W. McGovern, *Primitive Ideals*, Monty's article in the private atlas Wiki http:\\wiki.math.umd.edu\atlas\Primitive_ideals