On a family of nilpotent $K_{\mathbb{C}}$-orbits

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1. Introduction : Jordan algebras and degenerate principal series representations

Let me begin with a homage to Siddhartha, whom I pray will not be too offended by my distillation of a decade or so of work by him and his collaborators.

Let N be a simple real Jordan algebra. Then following a construction due to Tits, Kantor, and Koecher, one can attach to N a reductive Lie group G (the *conformal group* of N) . This is done in three stages. First, one forms the *Lorentz group* L of N, the subgroup of $GL(N)$ that preserves the Jordan norm up to scalars. Then one forms the *Poincare group* P of N as the semidirect product $P = LN$, and finally sets G as the group generated by P and *conformal inversion*: $x \to -x^{-1}$. One obtains in this way a (reductive) Lie group G with a parabolic $P = LN$ such that

(i) the nilradical N of P is abelian
(ii) P is conjugate to $\overline{P} = \theta(P)$

Vice-versa, starting with a parabolic subgroup P satisfying (i) and (ii), one can endow the nilradical N with the structure of a simple real Jordan algebra.

In this setting, let $\mathfrak{n}$ be the Lie algebra of N and let ν be the positive character of L corresponding to the determinant of the adjoint action of L on $\mathfrak{n}$ and consider the family of spherical principal series representations

$$I(s) = Ind_{LN}^{G}(\nu^{s} \otimes 1)$$

Then (i) implies that the K-types of $I(s)$ all have multiplicity 1, and (ii) implies that each irreducible constitutent of $I(s)$ carries an invariant hermitian form $\langle \, , \, \rangle_s$.The underlying Jordan algebra structure of this setting provides two additional tools for studying the reducibility of the representations $I(s)$ and the unitarizability of their irreducible constituents. First of all, associated to the primitive idempotents of N is a certain sequence of strongly orthogonal roots, the Cayley transforms of which provide an integral basis $\{\gamma_1, \ldots, \gamma_n\}$ for the K-types of $I(s)$:

(iii) $I(s)|_{K\text{-finite}} = \bigoplus_{\lambda \in \Lambda} V_\lambda$ where

$$\Lambda = \{\lambda = a_1\gamma_1 + \cdots + a_n\gamma_n \mid a_i \in \mathbb{Z} \quad \text{and } \lambda \text{ dominant}\}$$

Lastly, the Jordan norm on N gives rise to a family of differential intertwining operators

(iv) $D_m : I(-m) \to I(m)$

which Sahi employed to explicitly determine the *transition functions* between neighboring K-types

$$\begin{aligned} c_{\mu,i}(s) &= \langle x_i v_\mu, v_{\mu+\gamma_i} \rangle_s \quad , \quad x_i \in \mathfrak{p}_{\gamma_i} \quad , \quad v_\mu \in V_\mu \quad , \quad v_{\mu+\gamma_i} \in V_{\mu+\gamma_i} \\ d_{\mu,i}(s) &= \langle v_\mu, \overline{x_i} v_{\mu+\gamma_i} \rangle_s \quad , \quad \overline{x_i} \in \mathfrak{p}_{-\gamma_i} \end{aligned}$$

from both reduction points (the values of s where irreducible constituents of $I(s)$ percipitate) and the signature character of $I(s)$ could be deduced.[1]

In this talk, I intend to *reverse-engineer* all the remarkable circumstances of these representations in the context of a general semisimple Lie group. The crux of the matter, we shall see, is to replicate that peculiar sequence of strongly orthogonal roots.

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[1]That is to say, Sahi was able to determine the scalars s_λ such that

$$\langle \, , \, \rangle_{I(s)} = \bigoplus_{\lambda \in \Lambda} s_\lambda \langle \, , \, \rangle_{L^2(K)}|_{V_\lambda} \quad .$$

2. Sequences of strongly orthogonal noncompact weights

Let G be a connected semisimple Lie group, $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ a Cartan decomposition of the complexified Lie algebra of G, with θ the corresponding Cartan involution. We fix Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ and extend $\mathfrak{t}$ to a maximal compact Cartan subalgebra $\mathfrak{h}=\mathfrak{t}+\mathfrak{a}_c$ of $\mathfrak{g}$. We can and do fix a θ-stable positive system $\Delta^+(\mathfrak{h},\mathfrak{g})$, which upon restriction to$\mathfrak{k}$ yields a positive system $\Delta^+(\mathfrak{t},\mathfrak{k})$ for the roots of $\mathfrak{t}$ in $\mathfrak{k}$.

We now construct sequences $\Gamma=\{\gamma_1,\gamma_2,\dots,\gamma_r\}$ strongly orthogonal noncompact weights via the following prescription:

- γ_1 is the highest weight of an irreducible representation of K on $\mathfrak{p}$;[2]
- successive elements γ_i of the sequence are chosen so that
  - γ_i is an extremal weight of the representation of K on $\mathfrak{p}$
  - γ_i is strongly orthogonal to γ_j, $j=1,\dots,i-1$ (meaning the $\mathfrak{t}$-weight spaces $\mathfrak{g}_{\gamma_i\pm\gamma_j}$ are empty).
  - $\mu_i=\gamma_1+\cdots+\gamma_i$ is a dominant weight for $\Delta^+(\mathfrak{t},\mathfrak{k})$.

Examples 2.1.

- $G=SL(n,\mathbb{R})$, $K=SO(n)$

  In terms of the fundamental weights of K

$$\Gamma=\begin{cases}\Gamma=\{[2,0,\dots,0],[-2,2,0,\dots,0],\cdots,[0,\dots,0,-2,2,2],[0,\dots,0,\pm2,\mp2]\} & \text{if } n=2k\\ \Gamma=\{[2,0,\dots,0],[-2,2,0,\dots,0],\cdots,[0,\dots,0,-2,2,2],[0,\dots,0,-2,4]\} & \text{if } n=2k+1\end{cases}$$

  The Cayley transform of Γ coincides with the Kostant cascade of $\mathfrak{sl}_n$.
- $G=Sp(n,\mathbb{R})$, $K=U(n)$

$$\Gamma_+=\{[2,0,\dots,0],[-2,2,0,\dots,0],[0,-2,2,0,\dots,0],\dots,[0,\dots,0,-2,2],[0,\dots,0,-2]\}$$
$$\Gamma_-=\{\{[0,\dots,2],[0,\dots,0,2,-2],[0,\dots,0,-2,2,0,],\dots,,[2,-2,0,\dots,0].[-2,0,\dots,0]\}\}$$

  where Γ_+ consists of roots in $\mathfrak{p}_+$ and Γ_- consists of roots in $\mathfrak{p}_-$. These sequences correspond to Harish-Chandra sequences of strongly orthogonal roots for $\mathfrak{g}$ of hermitian symmetric type.
- $SL(n,\mathbb{H})$, $K=Sp(n)$

$$\Gamma=\{[0,2,0,\dots,0],[0,-2,0,2,0\dots,0],\dots,[0,\dots,0,-2,0,2]\}$$

  In this case, the extremal $\mathfrak{p}$ weights correspond to complex roots of $\mathfrak{g}$. This is the only case where this happens for simple Lie groups of classical type.
- $Sp(p,q)$, $K=Sp(p)\times Sp(q)$

$$\Gamma=\{[1,0,\dots,0;1,0.\dots,0],[-1,1,0,\dots,0;-1,1,0,\dots,0],\dots,[0,\dots,0,-1,1;0,\dots,0,-1,1,0,\dots,0]\}$$

  This cascade is completely disjoint from the Kostant cascade for $\mathfrak{sp}(p+q)$. Regarded as roots of $\mathfrak{g}$, these are all short roots.

3. A family of spherical nilpotent orbits attached to a SSONCW

Let $\Gamma=\{\gamma_1,\dots,\gamma_r\}$ be a maximal sequence of strongly orthogonal noncompact weights as constructed above. For each i, $1\le i\le r$, we can choose weight vectors $x_i\in\mathfrak{p}_{\gamma_i}$, $y_i\in\mathfrak{p}_{-\gamma_i}$ such that $y_i=\overline{x_i}$, and $h_i\in\mathfrak{t}$ such that

$$[x_i,y_i]=h_i\quad,\quad[h_i,x_i]=2x_i\quad,\quad[h_i,y_i]=-2y_i$$

That is, to each γ_i we can attach a normal S-triple $\mathfrak{s}_i$ in $\mathfrak{g}$. Moreover, since the γ_i are strongly orthogonal, if we set

$$\mathfrak{s}_i=span_{\mathbb{C}}(x_i,h_i,y_i)$$

then

$$[\mathfrak{s}_i,\mathfrak{s}_j]=0 \text{ if } i\neq j.$$

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[2]When G is not of hermitian-symmetric type, K act irreducibly on $\mathfrak{p}$. In this case, let γ_1 be the highest weight of the representation of K on $\mathfrak{p}$. If G is of hermitian symmetric type, $\mathfrak{p}$ decomposes into a direct sum of two irreducible representations of K

$$\mathfrak{p}=\mathfrak{p}_++\mathfrak{p}_-$$

(corresponding the holomorphic and anti-holomorphic tangent spaces of G/K); in this case, let γ_1 be the highest weight of the representation of K on $\mathfrak{p}_+$.

We set

$$X_i = x_1 + \cdots + x_i$$
$$H_i = h_1 + \cdots + h_i$$
$$Y_i = y_1 + \cdots + y_i$$

We obtain a family of *quasi-principal* normal S-triples. Finally, we set

$$\mathcal{O}_i = K_{\mathbb{C}} \cdot Y_i \qquad i = 1, \ldots, r$$

to obtain a telescoping family

$$\{0\} \equiv \mathcal{O}_0 \subset \overline{\mathcal{O}_1} \subset \overline{\mathcal{O}_2} \subset \cdots \subset \overline{\mathcal{O}_r}$$

of nilpotent $K_{\mathbb{C}}$-orbits.

Theorem 3.1. *Let* $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ *denote the ring of regular functions on the Zariski closure of the orbit* $\mathcal{O}_i$*. Then as a* K*-module*

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] = \bigoplus_{\lambda \in \Lambda_j} V_\lambda$$

where

$$\Lambda_j = \{\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i \mid a_j \in \mathbb{Z} \quad \textit{and } \lambda \textit{ dominant}\}$$

Sketch of Proof. Let $\mathfrak{t}_i = span_{\mathbb{C}}(h_1, \ldots, h_i)$, and let $\mathfrak{m}_i$ be the subalgebra of $\mathfrak{k}$ generated by root vectors $k_\alpha \in \mathfrak{k}_\alpha$ such that $\left.\alpha\right|_{\mathfrak{t}_i} = 0$. Then it happens that

$$\mathfrak{m}_i + \overline{\mathfrak{n}} \subset \mathfrak{k}^{Y_i}$$

Let $\mathbb{C}(\mathcal{O}_i)$ denote the ring of rational functions on $\mathcal{O}_i$. An algebraic Frobenious reciprocity argument then allows us to deduce that the multiplicity of K-type of highest weight λ will be zero if unless

$$\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$$

and moreover that each K-type V_λ appearing in $\mathbb{C}(\mathcal{O}_i)$ does so with multiplicity one. The observation that

$$k_i = \exp(i\pi h_i) \in K^{Y_i}$$

leads to the integrality condition on the coefficients $a_1, \ldots, a_i$.

At this point one knows that

$$\mathbb{C}\left[\overline{\mathcal{O}_i}\right] \subseteq \mathbb{C}(\mathcal{O}_i) = \bigoplus_{\lambda \in \Lambda_j} V_\lambda$$

To finish the proof, one shows that for each $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i \in \Lambda_i$, there exists a harmonic monomial in $S^{|a|}(\mathfrak{p})$ of weight λ that does not vanish at Y_i. Thus, accounting for each K-type of $\mathbb{C}(\mathcal{O})$ as a K-type in $\mathbb{C}\left[\overline{\mathcal{O}_i}\right]$ we are done.

4. Examples

(With thanks to John Stembridge, whose Posets Maple package enabled me to figure out the Hasse diagrams of the orbits).

4.1. $SL(n, \mathbb{R})$.

$$\begin{tikzcd}
\mathcal{O}^{I}_{(2)^{[n/2]}} \arrow[dr, Rightarrow] & & \mathcal{O}^{II}_{(2)^{[n/2]}} \arrow[dl, Rightarrow] \\
& \mathcal{O}_{(2)^{[n/2]-1}(1)^2} \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(2)^{[n/2]-2}(1)^4} \arrow[d, Rightarrow] & \\
& \vdots \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(2)(1)^{n-2}} \arrow[d, Rightarrow] & \\
& \mathcal{O}_{(1)^n} &
\end{tikzcd}$$

n even

$$\begin{tikzcd}
\mathcal{O}_{(2)^{[n/2]}} \arrow[d, Rightarrow] \\
\mathcal{O}_{(2)^{[n/2]-1}(1)^2} \arrow[d, Rightarrow] \\
\mathcal{O}_{(2)^{[n/2]-2}(1)^4} \arrow[d, Rightarrow] \\
\vdots \arrow[d, Rightarrow] \\
\mathcal{O}_{(2)(1)^{n-2}} \arrow[d, Rightarrow] \\
\mathcal{O}_{(1)^n}
\end{tikzcd}$$

n odd

4.2. $SU(2,q)$.

$$
\begin{tikzcd}[column sep=small]
 & & \mathcal{O}_{(-3)^2(-1)^{q-4}} \arrow[dll] \arrow[drr] & & \\
\mathcal{O}_{(-3)(-2)(-1)^{q-3}} \arrow[dd] \arrow[drr] & & & & \mathcal{O}_{(-3)(+2)(-1)^{q-3}} \arrow[dll] \arrow[dd] \\
 & & \mathcal{O}_{(-3)(+1)(-1)^{q-1}} \arrow[d] & \mathcal{O}_{(+3)(-1)^{q-1}} \arrow[dl] & \\
\mathcal{O}_{(-2)^2(-1)^{q-2}} \arrow[dr, Rightarrow] & & \mathcal{O}_{(+2)(-2)(-1)^{q-2}} \arrow[dl] \arrow[dr] & & \mathcal{O}_{(+2)^2(-1)^{q-2}} \arrow[dl, Rightarrow] \\
 & \mathcal{O}_{(-2)(+1)(-1)^{q-1}} \arrow[dr, Rightarrow] & & \mathcal{O}_{(+2)(+1)(-1)^{q-1}} \arrow[dl, Rightarrow] & \\
 & & \mathcal{O}_{(+1)^2(-1)^{q}} & &
\end{tikzcd}
$$

4.3. $SL(n, \mathbb{H})$.

$$
\begin{array}{ccc}
\mathcal{O}_{(2)^{[n/2]}} & \qquad\qquad & \mathcal{O}_{(2)^{[n/2]}(1)} \\
\Downarrow & & \Downarrow \\
\mathcal{O}_{(2)^{[n/2]-1}(1)^2} & & \mathcal{O}_{(2)^{[n/2]-1}(1)^3} \\
\Downarrow & & \Downarrow \\
\vdots & & \vdots \\
\Downarrow & & \Downarrow \\
\mathcal{O}_{(2)(1)^{n-2}} & & \mathcal{O}_{(2)(1)^{n-2}} \\
\Downarrow & & \Downarrow \\
\mathcal{O}_{(1)^n} & & \mathcal{O}_{(1)^n} \\
 & & \\
n \text{ even} & & n \text{ odd}
\end{array}
$$

4.4. $SO\,(2,p) \quad ; \quad p>4$.

$$
\begin{tikzcd}
 & & \mathcal{O}_{(-3)^2(-1)^{p-4}} \arrow[d] & & \\
\mathcal{O}^{I}_{(+3)(-1)^{p-1}} \arrow[dr, Rightarrow] & & \mathcal{O}_{(-3)(+1)(-1)^{p-2}} \arrow[dl] \arrow[dr] & & \mathcal{O}^{II}_{(+3)(-1)^{p-1}} \arrow[dl, Rightarrow] \\
 & \mathcal{O}^{I}_{(+2)(-2)(-1)^{p-2}} \arrow[dr, Rightarrow] & & \mathcal{O}^{II}_{(+2)(-2)(-1)^{p-2}} \arrow[dl, Rightarrow] & \\
 & & \mathcal{O}_{(+1)^2(-1)^{p}} & &
\end{tikzcd}
$$

4.5. $SO^{*}(2n)$.

$$
\begin{tikzcd}[column sep=small]
\mathcal{O}_{[\frac{n}{2}],0} \arrow[dr, Rightarrow] & & \mathcal{O}_{[\frac{n}{2}]-1,1} \arrow[dl] \arrow[dr] & \cdots & \cdots & \cdots & \cdots & \cdots & \mathcal{O}_{1,[\frac{n}{2}]-1} \arrow[dl] \arrow[dr] & & \mathcal{O}_{0,[\frac{n}{2}]} \arrow[dl, Rightarrow] \\
& \mathcal{O}_{[\frac{n}{2}]-1,0} \arrow[ddrr, Rightarrow, dotted] & \cdots & & & & & & & \mathcal{O}_{0,[\frac{n}{2}]-1} \arrow[ddll, Rightarrow, dotted] & \\
& & & & & \mathcal{O}_{(3)(1)^{n-3}} \arrow[d] & & & & & \\
& & & \mathcal{O}_{2,0} \arrow[dr, Rightarrow] & & \mathcal{O}_{1,1} \arrow[dl] \arrow[dr] & & \mathcal{O}_{0,2} \arrow[dl, Rightarrow] & & & \\
& & & & \mathcal{O}_{1,0} \arrow[dr, Rightarrow] & & \mathcal{O}_{0,1} \arrow[dl, Rightarrow] & & & & \\
& & & & & \mathcal{O}_{0,0} & & & & &
\end{tikzcd}
$$

where

$$\mathcal{O}_{r,s} = \mathcal{O}_{(+2)^{r}(-2)^{s}(1)^{n-2r-2s}}$$

4.6. $Sp\,(n,\mathbb{R})$.

$$\begin{tikzcd}
\mathcal{O}_{n,0} \arrow[dr, Rightarrow] & & \mathcal{O}_{n-1,1} \arrow[dl] \arrow[ddrr, dotted] & \cdots & \cdots & \cdots & \mathcal{O}_{1,n-1} \arrow[dr] \arrow[ddll, dotted] & & \mathcal{O}_{0,n} \arrow[dl, Rightarrow] \\
& \mathcal{O}_{n-1,0} \arrow[ddrr, Rightarrow, dotted] & \cdots & & \cdots & & & \mathcal{O}_{0,n-1} \arrow[ddll, Rightarrow, dotted] & \\
& & & & \mathcal{O}_{1,1} \arrow[dl] \arrow[dr] & & & & \\
& & & \mathcal{O}_{1,0} \arrow[dr, Rightarrow] & & \mathcal{O}_{0,1} \arrow[dl, Rightarrow] & & & \\
& & & & \mathcal{O}_{0,0} & & & &
\end{tikzcd}$$

where

$$\mathcal{O}_{r,s} = \mathcal{O}_{(+2)^r(-2)^s(+1)^{n-r-s}(-1)^{n-r-s}}$$

4.7. $Sp(p,q) \quad p \leq q$.

$$
\begin{tikzcd}
 & \mathcal{O}_{(+2)^p} \arrow[d, Rightarrow] & \\
 & \mathcal{O}_{(+2)^{p-1}(+1)(-1)} \arrow[d, Rightarrow, dotted] & \\
\mathcal{O}_{(+3)(+2)(+1)^{p-3}(-1)^{q-2}} \arrow[d] \arrow[dr] & \mathcal{O}_{(+2)^3(+1)^{p-3}(-1)^{q-3}} \arrow[d, Rightarrow] & \mathcal{O}_{(-3)(+2)(+1)^{p-2}(-1)^{q-3}} \arrow[d] \arrow[dl] \\
\mathcal{O}_{(+3)(+1)^{p-2}(-1)^{q-1}} \arrow[dr] & \mathcal{O}_{(+2)^2(+1)^{p-2}(-1)^{q-2}} \arrow[d, Rightarrow] & \mathcal{O}_{(-3)(+1)^{p-1}(-1)^{q-2}} \arrow[dl] \\
 & \mathcal{O}_{(+2)(+1)^{p-1}(-1)^{q-1}} \arrow[d, Rightarrow] & \\
 & \mathcal{O}_{(+1)^p(-1)^q} &
\end{tikzcd}
$$

5. Reprise: A family of degenerate principal series representations

I will now show how to attach to these orbits a family of degenerate principal series representations *endowed with* all the special circumstances that allowed Siddhartha's analysis of the Jordan algebra situation to proceed; that is to say, I shall display a family $I(s)$ of parabolically induced representations such that

(i) the K-types of $I(s)$ are

$$I(s)|_{K\text{-fiinte}} = \mathbb{C}\left[\overline{\mathcal{O}_i}\right] = \bigoplus_{\lambda\in\Lambda_j} V_\lambda$$

where $\Lambda = \{\lambda = a_1\gamma_1 + \cdots + a_n\gamma_n \mid a_i \in \mathbb{Z} \quad \text{and } \lambda \text{ dominant}\}$

each occurring with multiplicity one;

(ii) $I(s)$ carries an invariant Hermitian form;

amd

(iii) a family of differential intertwining operators $D_m : I(s) \to I(s+m)$.

5.1. Cayley transform. From here on we restrict our attention to the "quasi-principal" TDS $\{X, H, Y\} = \{X_n, H_n, Y_n\}$ and the big orbit $K_{\mathbb{C}} \cdot Y$. Set

$$c = Ad\left(\exp\left(\frac{i\pi}{4}(X+Y)\right)\right)$$

and

$$\begin{aligned}
\widetilde{X} &= c(X) = \frac{1}{2}(X+Y-iH)\\
\widetilde{H} &= c(H) = -i(X-Y)\\
\widetilde{Y} &= c(Y) = \frac{1}{2}(X+Y+iH)
\end{aligned}$$

then $\left\{\widetilde{X}, \widetilde{H}, \widetilde{Y}\right\}$ is a Cayley triple in $\mathfrak{g}_{\mathbb{R}}$, that is to say, a standard triple in $\mathfrak{g}_{\mathbb{R}}$ such that

$$\begin{aligned}
\theta\widetilde{X} &= -\widetilde{Y}\\
\theta\widetilde{Y} &= -\widetilde{X}\\
\theta\widetilde{H} &= -\widetilde{H}
\end{aligned}$$

5.2. Spherical induced representation. Since $\widetilde{H}$ is a semisimple element of $\mathfrak{p}_{\mathbb{R}}$, we can use it to construct a certain parabolic subgroup $P = MAN$ of G as well as a certain character of A. This construction goes as follows.

We define

$$\begin{aligned}
\mathfrak{n} &= \text{direct sum of positive eigenspaces of } ad\left(\widetilde{H}\right) \text{ in } \mathfrak{g}_{\mathbb{R}} \quad ,\\
\mathfrak{l} &= 0\text{-eigenspace of } ad\left(\widetilde{H}\right) \text{ in } \mathfrak{g}_{\mathbb{R}} \quad ,\\
\mathfrak{a} &= Z(\mathfrak{l}) \cap \mathfrak{p}_{\mathbb{R}} \quad ,\\
\mathfrak{m} &= \text{orthogonal complement of } \mathfrak{a} \text{ in } \mathfrak{l}
\end{aligned}$$

and then set

$$\begin{aligned}
M &= Z_K(\mathfrak{a})\exp(\mathfrak{m}) \quad ,\\
A &= \exp(\mathfrak{a}) \quad ,\\
N &= \exp(\mathfrak{n}) \quad .
\end{aligned}$$

Then $P = MAN$ is a (Langlands decomposition of a) parabolic subgroup of G. Now let ν be the element of the real dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ such that

$$\nu(h) = B_0\left(\widetilde{H}, h\right) \qquad \forall\ h \in \mathfrak{a}_0 \quad ,$$

where $B_0(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}_{\mathbb{R}}$ restricted to $\mathfrak{a}$.

Lemma 5.1. *The K-types of*

$$I(s) = Ind_{MAN}^{G}\left(1 \otimes e^{\nu s} \otimes 1\right)$$

coincide with those of $\mathbb{C}\left[\overline{K_{\mathbb{C}} \cdot Y}\right]$. In particular, if V is the realization of $I(s)$ in the compact picture then

$$V = \bigoplus_{\mu \in \mathcal{S}} V_\mu$$

with each K-type occuring with multiplicity one and in the $\mathbb{Z}$-span of the weights $\gamma_i \in \Gamma$.

Proof. This is essentially a verification that the algebraic Frobenious reciprocity argument used to determine the K-types in $\mathbb{C}\left[\overline{K_{\mathbb{C}} \cdot Y}\right]$ is compatible with the analytic Frobenious reciprocity argument used to identify the K-types of $I(s)$. The key to this is the observation that $\mathfrak{m}$ is preserved by the Cayley transform.

■ This lemma provides us with a replication of circumstances (i) and (iii).

To replicate circumstance (ii), we consider

$$w = \exp\left(\frac{\pi}{2}\left(\widetilde{X} - \widetilde{Y}\right)\right) \in K \quad .$$

It then happens that

Lemma 5.2. *With $P = MAN$, $\nu \in \mathfrak{a}^*$ and $w \in N_K(\mathfrak{a})$ defined as above, we have*

- $w \in N_K(\mathfrak{a})$;
- $wPw^{-1} = \overline{P}$, *the parabolic opposite to P*
- $Ad^*(w)\,\nu = -\nu$

I now quote a fundamental result of Knapp and Zuckerman

Theorem 5.3 (Knapp-Zuckerman)**.** *Suppose*

$$w \in N_K(\mathfrak{a}) \quad , \quad wPw^{-1} = \overline{P} \quad , \quad Ad^*(w)\,\nu = -\nu$$

then the Langlands quotient of $Ind_{MAN}^{G}(1 \otimes e^{\nu} \otimes 1)$ carries a non-degenerate hermitian form.

Since for generic s spherically induced representations are irreducible, Lemma 5.1, Theorem 5.3, and a Jantzen filtration argument gives us a invariant non-degenerate hermitian form on each irreducible submodule of

$$I(s) = Ind_{MAN}^{G}\left(1 \otimes e^{s\nu} \otimes 1\right) \quad .$$

■ We thus arrive at a replication of circumstance (ii). Well, almost. In our generalized setting, the space of K-finite vectors of $I(s)$ need not (e.g. when $G = SL(2k+1,\mathbb{R})$) be multiplicity free as a K-module. Nevertheless, at the reduction points we expect the unique irreducible quotients to be multiplicity-free, as their associated varieties should be one of the orbits $\overline{\mathcal{O}}_i$ for some $1 \leq i < n$.

Lastly, we turn to task of replicating circumstance (iv) in our generalized setting. Of course, our goal is not so much to replicate the Jordan algebraic Capelli operator *per se*, but rather to find a representation theoretical construct that provides the same functionality. And here again, once we figure out what we're looking for, we find it sitting right in our hands.

So what are we looking for? Well, ostensibly, the Kostant-Sahi Capelli operators are born from the Jordan norm on $\mathfrak{n}$. This form provides a homogeneous polynomial on $\mathfrak{n}$ which by duality leads to a certain constant coefficient operator on $C^\infty(\mathfrak{n})$. Interpreting the latter as the representation space for $I(s)$ in the noncompact picture, this operator becomes a certain L-quasi-invariant differential operator intertwining $I(1)$ with $I(-1)$. Cayley transforming to the compact picture, yields a K-invariant operator on K/M that continues to intertwine $I(1)$ and $I(-1)$. Then, via the Harish-Chandra homomorphism, Kostant and Sahi obtain the formula for the eigenvalue of D on a K-type V_α. That seems a lot to ask for.

However, by exploiting the natural duality between spherical principal series representations and generalized Verma modules of scalar type (corresponding to differentiating the functions in $Ind_{MAN}^{G}(1 \otimes e^{\nu} \otimes 1)$ at the identity), we can transfer the problem of finding differential operators intertwining the representations $I(s)$ to the problem of finding intertwining maps between generalized Verma modules of scalar type.

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{n} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$, and suppose now that E and E' are two 1-dimensional $\mathfrak{q}$-modules, having non-zero elements e and e' of weight λ and λ' with respect to the $\mathfrak{a}$-part of $\mathfrak{q}$. Then every $\mathfrak{g}$-homomorphism

$$\phi : U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} E' \quad \rightarrow \quad U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} E$$

is determined by the image $\phi(1 \otimes e')$. Via the Poincare-Birkhoff-Witt theorem we can write

$$\phi(1 \otimes e') = u \otimes e$$

for some $u \in U(\overline{\mathfrak{n}})$. Since E and E' are 1-dimensional and $\mathfrak{m}$ is semisimple, it follows that $u \in U(\overline{\mathfrak{n}})^{\mathfrak{m}}$. Morever, since ϕ must be in particular an $\mathfrak{a}$-homomorphism, u must have weight $\lambda' - \lambda$. Hence, u must be an $\mathfrak{l}$-semi-invariant of weight $\lambda' - \lambda$ in $U(\overline{\mathfrak{n}})$. Moreover, u has to be annihilated by $\mathfrak{n}$ since $1 \otimes e'$ is annihilated by $\mathfrak{n}$. Finally, just as in the case of ordinary Verma modules (in fact, it follows from this case since every generalized Verma module is a quotient of an ordinary Verma module), $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E'$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} E$ have to have the same infinitesimal character, and this leads to the stipulation that there exist a Weyl group element such that

$$w(\lambda + \rho) = \lambda' + \rho \tag{***}$$

How are we going to find such a u?

Lemma 5.4. *Let $r \in \frac{1}{2}\mathbb{N}$ be defined by*

$$\frac{1}{2} \sum_{\alpha \in \Sigma(\mathfrak{t}_1, \mathfrak{g})} m_\alpha \alpha = r(\gamma_1 + \cdots + \gamma_n)$$

Then, for any integer k, $I(2k + r)$ contains a irreducible finite-dimensional spherical representation of $\mathfrak{g}$ of highest weight $k(\gamma_1 + \cdots + \gamma_n)$.

Proof. In Helgason, *Groups and Geometric Analysis*, one can find the following statement.[3]

Theorem 5.5. *Let π be an irreducible finite-dimensional representation of G. Then the following statements are equivalent.*

(i) *π has a non-zero K-fixed vector.*
(ii) *The highest weight ν of π vanishes on $\mathfrak{t}_0 \subset \mathfrak{m}$, and the restriction of ν to $\mathfrak{a}$ is such that*

$$\frac{\langle \nu, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \textit{ for every restricted root } \beta \in \mathbb{Z}$$

It is easy to see that the weights $k\mu_n = k(\gamma_1 + \cdots + \gamma_n)$ satisfy the conditions (ii), and so there exists a finite-dimensional spherical representation of highest weight $k\mu_n$. Moreover, if ϕ_ν is the highest weight vector for such a representation π_k and ϕ_K is the corresponding spherical vector, it is easy to see that the matrix element functions

$$\phi(g) = \langle \pi(g) \phi_\nu, \phi_K \rangle$$

transform as the irreducible finite-dimensional representation contragredient to π, and moreover

$$\phi(g) \in I(2k + r) \quad .$$

These last two statements follow from easy calculations that are carried out very explicitly in Knapp's book (Representation Theory of Semisimple Groups, §9.6) □ It is easy to see that in the noncompact picture this correspond to a certain $\mathfrak{g}$-invariant set of polynomials on $\overline{\mathfrak{n}}$. Moreover, F_μ is spherical, and its highest weight vector ψ is $\mathfrak{m}$-invariant and $\mathfrak{l}$-semi-invariant. It follows that the image of ψ in $U(\mathfrak{n})$ via the symmeterizer map will have all the properties we need for u except possibly (***). But it's just as easy to check that (***) holds as well - using the same Weyl group element w we used to establish the existence of the hermitian form on $I(s)$.

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[3] pg. 535.

6. Example: $SL(2k+1,\mathbb{R})$

We begin by choosing

$$\mathfrak{t} = \begin{bmatrix} 0 & \cdots & & 0 & t_1 \\ 0 & & & t_2 & 0 \\ \vdots & & & & \vdots \\ 0 & -t_2 & 0 & \cdots & 0 \\ -t_1 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

as a maximal compact subalgebra of $\mathfrak{k} \approx \mathfrak{so}(2k+1)$. From this choice, we are led to

$$X = \left[\begin{array}{c|c|c} \frac{i}{2}\mathbf{I}_k & \mathbf{0} & \frac{1}{2}\mathbf{J}_k \\ \hline \mathbf{0} & 0 & \mathbf{0} \\ \hline \frac{1}{2}\mathbf{J}_k & \mathbf{0} & -\frac{i}{2}\mathbf{I}_k \end{array}\right]$$

$$H = \left[\begin{array}{c|c|c} \mathbf{0}_k & \mathbf{0} & i\mathbf{J}_k \\ \hline \mathbf{0} & 0 & \mathbf{0} \\ \hline -i\mathbf{J}_k & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$Y = \left[\begin{array}{c|c|c} -\frac{i}{2}\mathbf{I}_k & \mathbf{0} & \frac{1}{2}\mathbf{J}_k \\ \hline \mathbf{0} & 0 & \mathbf{0} \\ \hline \frac{1}{2}\mathbf{J} & \mathbf{0} & \frac{i}{2}\mathbf{I}_k \end{array}\right]$$

where $\mathbf{I}_k$ is the $k \times k$ identity matrix and

$$\mathbf{J}_k = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

Upon applying the Cayley transform we arrive at the following S-triple in $\mathfrak{g}_{\mathbb{R}}$:

$$\widetilde{X} = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{J}_k \\ \hline \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

$$\widetilde{H} = \left[\begin{array}{c|c|c} \mathbf{I}_k & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & -\mathbf{I}_k \end{array}\right]$$

$$\widetilde{Y} = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & 0 & \mathbf{0} \\ \hline \mathbf{J}_k & \mathbf{0} & \mathbf{0} \end{array}\right]$$

The parabolic constructed from $\widetilde{H}$ is of the form

$$P = \left[\begin{array}{c|c|c} \mathbf{L} & \mathbf{p} & \mathbf{n}_2 \\ \hline \mathbf{0} & -2tr(\mathbf{L}) & \mathbf{q} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{L} \end{array}\right]$$

and the element $w = \exp\left(\frac{\pi}{2}(\widetilde{X}+\widetilde{Y})\right) \in N_K(\mathfrak{a})$ is

$$w = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \mathbf{I}_k \\ \hline \mathbf{0} & 1 & \mathbf{0} \\ \hline -\mathbf{I}_k & \mathbf{0} & \mathbf{0} \end{array}\right]$$