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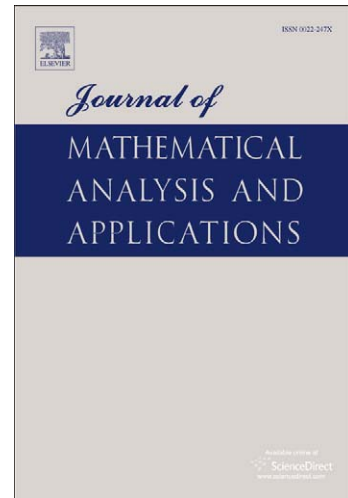
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ON THE EVALUATION OF SOME SELBERG-LIKE INTEGRALS

B. BINEGAR

ABSTRACT. Several methods of evaluation are presented for a family $I_{n,d,p}$ of Selberg-like integrals that arise in the computation of the algebraic-geometric degrees of a family of spherical nilpotent orbits associated to the symmetric space of a simple real Lie group. Adapting the technique of Nishiyama, Ochiai and Zhu, we present an explicit evaluation in terms of certain iterated sums over permutation groups. The resulting formula, however, is only valid when the integrand involves an even power of the Vandermonde determinant. We then apply, to the general case, the theory of symmetric functions and obtain an evaluation of the integral $I_{n,d,p}$ as a product of polynomial of fixed degree times a particular product of gamma factors; thereby identifying the asymptotics of the integrals with respect to their parameters. Lastly, we derive a recursive formula for evaluation of another general class of Selberg-like integrals, by applying some of the technology of generalized hypergeometric functions.

1. INTRODUCTION

In 1944, while still a high school student, Atle Selberg published the following multivariate generalization of Euler's beta integral formula:

$$(1.1) \quad \int_{[0,1]^n} \left(\prod_{i=1}^n (x_i)^{r-1} (1-x_i)^{s-1} \right) \left(\prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\kappa} \right) d^n x \\ = \prod_{i=1}^n \frac{\Gamma(i\kappa + 1) \Gamma(r + (i-1)\kappa) \Gamma(s + (i-1)\kappa)}{\Gamma(\kappa + 1) \Gamma(r + s + 2 + (n-i-2)\kappa)}.$$

For some 35 years this result laid in deep hibernation - until it was rediscovered and vigorously reanimated by Askey [As], Macdonald [Mac1], Koranyi [Kor] and many others. Indeed, since its rediscovery, the Selberg formula has been generalized in several directions ([Ao], [Kad], [Kan], [Ri]), and has found important applications in both pure mathematics ([Kor],[FK], [KO], [NO], [Meh]) and physics ([Fo],[Ve]).

Still today the Selberg integrals lay at the heart of a fascinating nexus of representation theory, algebraic geometry, analysis, and combinatorics. A very nice illustration of this is found in a recent paper by K. Nishiyama and H. Ochiai [NO]. Most ostensibly, this paper deals with the problem of calculating the Bernstein degrees of singular highest weight representations of the metaplectic group. However,

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it turns out that the crux of the matter is to evaluate an integral of the form

$$(1.2) \quad \int_{\mathcal{S}_i} \left(\prod_{1 \leq j \leq i} x_j \right)^s \left(\prod_{1 \leq j < k \leq i} (x_j - x_k) \right)^d d^i x \quad .$$

This they do by reinterpreting the integral as an integral over the symmetric cone associated to the space of positive, real symmetric matrices. They are then able to use results of Faraut and Koranyi [FK] (which go back to the original Selberg formula via [Kor]) to obtain an explicit evaluation as a certain product of gamma functions of the parameters. The authors then remark that their computation of the Bernstein degree is equivalent to computing the algebraic-geometric degree of the determinantal varieties $Sym_n(m) = \{X \in M_n(\mathbb{C}) \mid {}^t X = X, \text{rank}(X) \leq m\}$, and that they have thereby reproduced the classical formulae of Giambelli.

Inspired by the methods and results of the current generation of Japanese representation theorists (e.g., [NOTYK], [NOZ], [KO]) we have in [B] derived a formula for the leading term of the Hilbert polynomials of a family of spherical nilpotent $K_{\mathbb{C}}$ -orbits associated to the symmetric space of a simple real Lie group. This formula, together with the restricted root data, reduces the problem of the determining the algebraic-geometric degree of (the projectization of) such an orbit to the evaluation of an integral that is either of the form (1.2), which happens only when G/K is Hermitian symmetric, or of the form

$$(1.3) \quad \int_{\mathcal{S}_i} \left(\prod_{1 \leq j \leq i} x_j \right)^s \left(\prod_{1 \leq j < k \leq i} (x_j^2 - x_k^2) \right)^d d^i x$$

and where, in both cases, \mathcal{S}_i is the domain

$$(1.4) \quad \mathcal{S}_i = \left\{ x \in \mathbb{R}^i \mid x_1 \geq x_2 \geq \cdots \geq x_i \geq 0 \quad , \quad \sum_{j=1}^i x_j \leq 1 \right\} \quad .$$

Let us remark here that integrals of the form (1.2), while replete with applications and interpretations (such as in [NO]), can also be evaluated simply by a change of variables ([Mac2] pg. 286) and an application of the original Selberg formula. On the other hand, integrals of the form (1.3) also have interpretations and applications outside our particular representation theoretical context. For example, integrals of this form arise quite naturally in the context of analysis on symmetric domains (see, e.g., [FK], pg. 197). Yet it seems, except for very particular values of the parameters, no explicit evaluation is known.

At first this may seem a modest oversight in the literature; as apparently the only difference between the two integrals is the replacement of the linear difference factors $x_i - x_j$ in (1.2) with the quadratic difference factors $x_i^2 - x_j^2$. Moreover, from a representation theoretical point of view, the difference/connection between the two integrals is just a reflection of the fact that the restricted root systems in the Hermitian symmetric case are of type $A_{n-1} \approx \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\}$, while, in the non-Hermitian symmetric situation, the restricted roots systems will involve both sums and differences (as well as multiples) of the standard basis vectors e_i . However, the substitution $x_i - x_j \rightarrow x_i^2 - x_j^2$, also leads to a breaking of a subtle $x_i \longleftrightarrow 1 - x_i$ symmetry in the original Selberg integral. This loss of symmetry at least partially explains why the integral (1.3) is such a bugger.

Henceforth, we shall denote by $I_{n,d,p}$ the integral

$$(1.5) \quad I_{n,d,p} \equiv \int_{\mathcal{S}_n} \left(\prod_{1 \leq i \leq n} x_i \right)^p \left(\prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2) \right)^d d^n x$$

where the domain \mathcal{S}_n is

$$(1.6) \quad \mathcal{S}_n = \left\{ x \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \cdots \geq x_n \geq 0, \quad \sum_{i=1}^n x_i \leq 1 \right\}$$

as above. Note that when d is an even integer, the integrand is a homogeneous symmetric polynomial and that (in any case) \mathcal{S}_n is a fundamental domain for the action of the symmetric group \mathfrak{S}_n . Because of these circumstances we can compute $I_{n,2k,p}$ as $n!$ times the integral of the same integrand over the simpler region

$$\Omega_n = \mathfrak{S}_n \cdot \mathcal{S}_n = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0; \sum_{i=1}^n x_i \leq 1 \right\} .$$

This observation and the method of Nishiyama, Ochiai and Zhu [NOZ] allows us to obtain in §2 the following result.

Theorem 1. *If d is even then*

$$I_{n,d,p} = \frac{1}{n!} \frac{1}{\Gamma(a+1)} \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \left[\text{sgn}(\sigma_1 \cdots \sigma_d) \times \prod_{i=1}^n \Gamma(2\sigma_1(i) + \cdots + 2\sigma_d(i) - 2d + p + 1) \right]$$

where

$$a \equiv n(p+1 + d(n-1)) .$$

Although this result is fairly explicit, it has two unpleasant features. First of all, it is valid only for even d (when d is odd the integrand is skew-symmetric and so the integral can not be computed by extending its domain to Ω_n). Secondly, the sums over permutations are extremely arduous to compute even for relatively small values for n and d . Toward the end of §2 we derive the following result which sheds a little more light on this valuation of $I_{n,d,p}$ for large values of the parameters:

Proposition 2.3. *Let $S_{n,d,p}$ be*

$$\sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \left[\text{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n \Gamma(2\sigma_1(i) + \cdots + 2\sigma_d(i) - 2d + p + 1) \right]$$

(i.e., $S_{n,d,p}$ is the sum on the right hand side of the formula in Proposition 1.3), then

$$S_{n,d,p} = \Phi_{n,d}(p) \prod_{i=1}^n \Gamma(p+1 + d(i-1))$$

where

$$\Phi_{n,d}(p) = \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \text{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n (p+1 + d(i+1))_{2\mu_i(\sigma)} .$$

Here $(n)_k = (n)(n-1)\cdots(n-k-1)$ is the usual Pochhammer symbol, and $\mu_i(\sigma) = \mu_i(\sigma_1, \dots, \sigma_d)$ is a particular integer-valued function on $(\mathfrak{S}_n)^d$ (corresponding roughly to the height of the i^{th} part of a partition constructed from the $\sigma_1, \dots, \sigma_d$ above its minimal possible value.) It is readily seen that the degree of the polynomial factor $\Phi_{n,d}(p)$ is $\leq \frac{d}{2}n(n-1)$. However, explicit computations reveal that this bound on the degree of $\Phi_{n,d}(p)$ is far from being sharp. This is to be remedied later in §3 (see Remark 3.1.)

The theory of symmetric functions provides another point of entry into the topic of generalized Selberg integrals and, in fact, much of the work on generalized Selberg integrals during the 1980's and 1990's ([Mac1], [Ao], [Kad], [Ri]) was predicated on this point of view. We have already noted that $I_{n,d,p}$ is an integral of a symmetric or skew-symmetric homogeneous polynomial over a fundamental domain for the natural action of the symmetric group \mathfrak{S}_n on \mathbb{R}^n . In fact, if we denote by e_n the n^{th} elementary symmetric function

$$e_n(\mathbf{x}) \equiv \prod_{i=1}^n x_i \quad ,$$

by $\Delta^{(n)}$ the Vandermonde determinant

$$\Delta^{(n)}(\mathbf{x}) = \det \left[x_i^{n-j} \right]_{i,j=1,\dots,n} = \sum_{1 \leq i < j \leq n} (x_i - x_j)$$

and by s_δ the Schur symmetric function corresponding to the staircase partition $\delta = (n-1, \dots, 1, 0)$

$$s_\delta(\mathbf{x}) \equiv \frac{\det \left[x_i^{n-j+\delta_j} \right]_{i,j=1,\dots,n}}{\det \left[x_i^{n-j} \right]_{i,j=1,\dots,n}} = \sum_{1 \leq i < j \leq n} (x_i + x_j) \quad ,$$

then then we can express the integral (1.5) as

$$I_{n,d,p} = \int_{\mathcal{S}_n} (e_n(\mathbf{x}))^p (s_\delta(\mathbf{x}))^d \left| \Delta^{(n)}(\mathbf{x}) \right|^d d\mathbf{x} \quad .$$

In §3 we use the theory of symmetric functions and an integral formula of Macdonalds, to arrive at the following formula valid for any positive integer d .

Theorem 2. *Let $I_{n,d,p}$ and $a = n(p+1+d(n-1))$.*

- *If d is even, then $I_{n,d,p}$ is of the form*

$$I_{n,d,p} = \frac{1}{\Gamma(a+1)} \Phi(p) \prod_{j=0}^{n-1} \Gamma(p+dj+1)$$

with $\Phi(p)$ a polynomial in p of degree $\leq \frac{d}{4}n(n-1)$.

- *If d is odd,*

$$I_{n,d,p} = \frac{1}{\Gamma(a+1)} \Phi(p) \prod_{j=0, \text{even}}^{n-1} \Gamma(p+dj+1) \prod_{j=1, \text{odd}}^{n-1} \Gamma(p+dj+1/2)$$

with $\Phi(p)$ a polynomial in p of degree $\leq \frac{d}{2}n(n-1) - \sum_{i=1}^n \left\lfloor \frac{d(n-i)+1}{2} \right\rfloor$.

Here $\lfloor \frac{k}{2} \rfloor$ denotes the integer part of $k/2$.

A third approach to the evaluation of Selberg type integrals is demonstrated in a remarkable paper by Kaneko [Kan] (which is in turn based on ideas of Aomoto [Ao]). Therein, the evaluation of a family of Selberg type integrals is carried out by identifying certain recursive formulae within the family, which in turn lead to certain holonomic systems of PDEs satisfied by the integrals. Kaneko then shows that the unique analytic solutions to these PDEs are expressible in terms of generalized hypergeometric series and thereby obtains explicit evaluations in terms of generalized hypergeometric functions.

In §4 we generalize the integrand of the original problem a bit and focus our efforts on obtaining recursive formulas for another general class of Selberg-like integrals. More explicitly, we consider integrals of the form

$$(1.7) \quad J_{n,\kappa}(\Phi_\lambda) \equiv \int_{\mathcal{S}_n} \Phi_\lambda(\mathbf{x}) \left(\Delta^{(n)}(\mathbf{x}) \right)^\kappa d\mathbf{x}$$

where $\{\Phi_\lambda\}$ is some basis for homogeneous symmetric polynomials indexed of partitions λ (e.g., the Φ_λ might be monomial symmetric functions, or Jack symmetric functions). We first introduce a change of variables and show how by a sequence of such transformations any integral of the form (1.7) can, in principle, be reduced to a sum of products of beta integrals. We then introduce generalized Taylor coefficients $c_{\lambda\mu}$ and generalized binomial coefficients $\binom{\lambda}{\mu}$, defined, respectively by the formulas

$$\Phi_\lambda(\mathbf{x}_{(n)}) = \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} c_{\lambda\mu} \Phi_\mu(\mathbf{x}_{(n-1)}) x_n^i$$

and

$$\Phi_\lambda(\mathbf{x}_{(n)} + t\mathbf{1}_{(n)}) = \sum_{\mu} \binom{\lambda}{\mu} \Phi_\mu(\mathbf{x}_{(n)}) t^{|\lambda|-|\mu|}$$

Here $\mathbf{x}_{(n-1)}$ denotes the vector in \mathbb{R}^{n-1} obtained from $\mathbf{x}_{(n)} = (x_1, \dots, x_n) \in \mathbb{R}^n$ by dropping the last coordinate x_n , and $\mathbf{1}_{(n)} = (1, \dots, 1) \in \mathbb{R}^n$. From these formulas we obtain, from our change of variables formula, the following recursive formula

Theorem 3.

$$J_{n,\kappa}(\Phi_\lambda) = \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} \sum_{\nu} (n)^{|\nu|-n-i-1} c_{\lambda\mu} \binom{\mu}{\nu} \\ \times B\left(|\lambda| + |\nu| + 1, n - 1 + \frac{\kappa}{2}n(n-1) + |\nu|\right) J_{n-1,\kappa}(\Phi_\nu) \quad .$$

2. AN EXPLICIT EVALUATION OF $I_{n,d,p}$ FOR THE CASE OF EVEN d

In this section we follow the method of Nishiyama, Ochiai and Zhu [NOZ] to find a closed expression for $I_{n,d,p}$ for case when d is an even integer.

We begin by noting that, for even d , the integrand is a symmetric polynomial in the variables x_i and that the region of integration

$$\mathcal{S}_n = \left\{ \mathbf{x} \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0 \quad , \quad \sum_{i=1}^n x_i \leq 1 \right\}$$

is a fundamental domain for the natural action of the symmetric group \mathfrak{S}_n on

$$\Omega_n = \mathfrak{S}_n \cdot \mathcal{S}_n = \left\{ \mathbf{x} \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, \dots, n, \ \text{and} \ \sum_{i=1}^n x_i \leq 1 \right\} .$$

Because of this we can write

$$I_{n,d,p} = \frac{1}{n!} \int_{\Omega_n} \left(\prod_i x_i^p \right) \left(\prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2)^d \right) d^n x .$$

Next, we set

$$a = n(p + d(n - 1) + 1)$$

and note that degree of the integrand is $a - n$.

We now use the identity

$$\Gamma(a + 1) \int_{\Omega_n} f(\mathbf{x}) d\mathbf{x} = \int_{\Omega_n \times (0, \infty)} f(\mathbf{x}) s^a e^{-s} d\mathbf{x} ds$$

and make a change of variables

$$\begin{aligned} y_i = s x_i & \iff x_i = \frac{y_i}{t + \sum_{k=1}^n y_k} \\ t = s(1 - \sum_{i=1}^n x_i) & \iff s = t + \sum_{k=1}^n y_k \end{aligned}$$

which maps $\Omega_n \times (0, \infty)$ diffeomorphically onto $(0, \infty)^n \times (0, \infty)$. The Jacobian of this transformation is easily seen to be

$$\frac{\partial(y, t)}{\partial(x, s)} = s^n = \left(t + \sum_{k=1}^n y_k \right)^n$$

and so we have, for any function f homogeneous of degree $a - n$,

$$\begin{aligned} \Gamma(a + 1) \int_{\Omega_n} f(\mathbf{x}) d\mathbf{x} &= \int_{\Omega_n \times (0, \infty)} f(\mathbf{x}) s^a e^{-s} d\mathbf{x} ds \\ &= \int_{(0, \infty)^n \times (0, \infty)} f\left(\frac{\mathbf{y}}{t + \sum_{k=1}^n y_k}\right) \left(t + \sum_{k=1}^n y_k\right)^a \\ &\quad \times e^{-t - \sum_{k=1}^n y_k} \left(t + \sum_{k=1}^n y_k\right)^n dy dt \\ &= \int_{(0, \infty)^n \times (0, \infty)} f(\mathbf{y}) e^{-t - \sum_{k=1}^n y_k} dy dt \\ &= \int_{(0, \infty)^n} f(\mathbf{y}) e^{-\sum_{k=1}^n y_k} dy . \end{aligned}$$

We thus arrive at

$$I_{n,d,p} = \frac{1}{n!} \frac{1}{\Gamma(a + 1)} \int_{(0, \infty)^n} \left(\prod_{i=1}^n y_i^p \right) \left(\prod_{1 \leq i < j \leq n} (y_i^2 - y_j^2)^d \right) e^{-\sum_{k=1}^n y_k} dy .$$

The next step is to expand the second product using the identity

$$\prod_{1 \leq i < j \leq n} (y_i^2 - y_j^2) = \det \left(y_j^{2(i-1)} \right)_{i=1, \dots, n} = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n y_i^{2(\sigma(i)-1)} .$$

We have

$$\begin{aligned} \left(\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n y_i^{2(\sigma(i)-1)} \right)^d &= \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} [\operatorname{sgn}(\sigma_1) \cdots \operatorname{sgn}(\sigma_d) \\ &\quad \times \prod_{i=1}^n y_i^{2(\sigma_1(i)-1) + \cdots + 2(\sigma_d(i)-1)}] \\ &= \sum_{\sigma \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n y_i^{2(\sum_{j=1}^d \sigma_j(i)) - 2d} \end{aligned}$$

and so

$$\begin{aligned} I_{n,d,p} &= \frac{1}{n!} \frac{1}{\Gamma(a+1)} \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \int_{(0,\infty)^n} \\ &\quad \times \prod_{i=1}^n y_i^{2(\sum_{j=1}^d \sigma_j(i)) - 2d + p} e^{-\sum_{k=1}^n y_k} dy \\ &= \frac{1}{n!} \frac{1}{\Gamma(a+1)} \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} [\operatorname{sgn}(\sigma_1 \cdots \sigma_d) \\ &\quad \times \prod_{i=1}^n \left(\int_0^\infty y_i^{2(\sum_{j=1}^d \sigma_j(i)) - 2d + p} e^{-y_i} dy_i \right)] \\ &= \frac{1}{n!} \frac{1}{\Gamma(a+1)} \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} [\operatorname{sgn}(\sigma_1 \cdots \sigma_d) \\ &\quad \times \prod_{i=1}^n \Gamma(2\sigma_1(i) + \cdots + 2\sigma_d(i) - 2d + p + 1)] \end{aligned}$$

where we have used the Euler's formula for the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad .$$

In summary,

Theorem 1. *If d is even and*

$$I_{n,d,p} \equiv \int_{\mathcal{S}_n} \left(\prod_i x_i^p \right) \left(\prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2)^d \right) d^n x$$

then

$$\begin{aligned} I_{n,d,p} &= \frac{1}{n!} \frac{1}{\Gamma(a+1)} \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} [\operatorname{sgn}(\sigma_1 \cdots \sigma_d) \\ &\quad \times \prod_{i=1}^n \Gamma(2\sigma_1(i) + \cdots + 2\sigma_d(i) - 2d + p + 1)] \end{aligned}$$

where $a = n(p + d(n-1) + 1)$.

We now focus our attention on the product of gamma factors

$$\prod_{i=1}^n \Gamma(2\sigma_1(i) + \cdots + 2\sigma_d(i) - 2d + p + 1) \quad .$$

We begin by forming the vector

$$\left[\sum_{j=1}^d \sigma_j(1), \sum_{j=1}^d \sigma_j(2), \dots, \sum_{j=1}^d \sigma_j(n) \right]$$

and then reordering the components in increasing order to form

$$\gamma(\sigma) \equiv \left[\min_i \left\{ \sum_{j=1}^d \sigma_j(i) \right\}, \dots, \max_i \left\{ \sum_{j=1}^d \sigma_j(i) \right\} \right] \quad .$$

Lemma 2.1. *For any arrangement $\sigma \in (\mathfrak{S}_n)^d$ we have*

$$\gamma_i(\sigma) \geq \frac{d}{2}(i+1) \quad .$$

Proof. We first note that

$$\gamma_1(\sigma) \geq d$$

follows readily from the requirement that each $\sigma_j(i) \geq 1$. Next, we note that for any arrangement $(\sigma_1, \dots, \sigma_d)$

$$\sum_{j=1}^d \sum_{i=1}^n \sigma_i(j) = \sum_{j=1}^d (1 + 2 + \cdots + n) = \frac{d}{2}n(n+1)$$

and that the particular arrangement where

$$\sigma_k = \begin{cases} [1, 2, \dots, n-1, n] & \text{if } k \text{ is even} \\ [n, n-1, \dots, 2, 1] & \text{if } k \text{ is odd} \end{cases}$$

leads to

$$\begin{aligned} \gamma(\sigma) &= \left[\frac{d}{2}(1) + \frac{d}{2}(n), \frac{d}{2}(2) + \frac{d}{2}(n-1), \dots, \frac{d}{2}(n) + \frac{d}{2}(1) \right] \\ &= \left[\frac{d}{2}(n+1), \frac{d}{2}(n+1), \dots, \frac{d}{2}(n+1) \right] \quad . \end{aligned}$$

Now note that one cannot decrease the last component of $\gamma(\sigma)$ further without violating the requirement that $\sum_{i=1}^n \gamma_i(\sigma) = \frac{d}{2}n(n+1)$ (and the stipulated ordering $\gamma_i(\sigma) \leq \gamma_{i+1}(\sigma)$). Thus, we have

$$\gamma_n(\sigma) \geq \frac{d}{2}(n+1) \quad .$$

Finally, we observe that the arrangement

$$\sigma_k = \begin{cases} [1, 2, \dots, i-1, i, i+1, \dots, n] & \text{if } k \text{ is even} \\ [i, i-1, \dots, 2, 1, i+1, \dots, n] & \text{if } k \text{ is odd} \end{cases}$$

leads to

$$\gamma(\sigma) = \left[\frac{d}{2}(i+1), \dots, \frac{d}{2}(i+1), d(i+1), \dots, d(n) \right]$$

and so, by essentially the reasoning as above,

$$\gamma_i(\sigma) \geq \frac{d}{2}(i+1) \quad .$$

□

Proposition 2.2. *Let*

$$S_{n,d}(p) \equiv \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n \Gamma(2\sigma_1(i) + \cdots + 2\sigma_d(i) - 2d + p + 1) \quad .$$

Then

$$S_{n,d}(p) = \Phi_{n,d}(p) \prod_{i=1}^n \Gamma(p + 1 + d(i-1))$$

where

$$(2.1) \quad \Phi_{n,d}(p) = \left(\sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n (p + 1 + d(i+1))_{2\mu_i(\sigma)} \right)$$

is a polynomial in p of degree $\leq \frac{d}{2}n(n-1)$.

Proof. Set

$$\mu_i(\sigma) = \gamma_i(\sigma) - \frac{d}{2}(i+1)$$

so that

$$\mu_i(\sigma) \geq 0 \quad , \quad i = 1, \dots, n \quad .$$

For each term of the iterated sum we can arrange the gamma factors so that their arguments are non-decreasing. In other words we can write

$$\begin{aligned} S_{n,d}(p) &= \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n \Gamma(2\gamma_i(\sigma) - 2d + p + 1) \\ &= \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n \Gamma(2\mu_i(\sigma) + d(i+1) - 2d + p + 1) \\ &= \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n \Gamma(p + 1 + d(i-1) + 2\mu_i(\sigma)) \quad . \end{aligned}$$

We now introduce the Pochhammer symbols $(k)_n$ defined by

$$(k)_n \equiv \frac{\Gamma(k+n)}{\Gamma(k)} \quad ,$$

noting that $(k)_0 = 1$ and that for positive integers n

$$(k)_n = (k)(k+1)\cdots(k+n-1) \quad .$$

Returning to our expression for $S_{n,d}(p)$ we have

$$\begin{aligned}
S_{n,d}(p) &= \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n \Gamma(p+1+2\mu_i(\sigma) + d(i-1)) \\
&= \sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n (p+1+d(i-1))_{2\mu_i(\sigma)} \Gamma(p+1+d(i-1)) \\
&= \left(\sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n (p+1+d(i-1))_{2\mu_i(\sigma)} \right) \\
&\quad \times \prod_{i=1}^n \Gamma(p+1+d(i+1)) \quad .
\end{aligned}$$

Finally we note that each factor

$$\prod_{i=1}^n (d(i-1) + p + 1)_{2\mu_i(\sigma)}$$

is a polynomial in p of total degree

$$\begin{aligned}
\sum_{i=1}^n 2\mu_i(\sigma) &= \sum_{i=1}^n (2\gamma_i(\sigma) - d(i+1)) \\
&= 2 \sum_{i=1}^n \gamma_i(\sigma) - d \left(\frac{1}{2}n(n+1) + n \right) \\
&= 2 \sum_{i=1}^n \sum_{j=1}^d \sigma_j(i) - d \left(\frac{1}{2}n(n+1) + n \right) \\
&= 2d \left(\sum_{i=1}^n i \right) - d \left(\frac{1}{2}n(n+1) + n \right) \\
&= 2d \frac{1}{2}n(n+1) - \frac{d}{2}n(n+1) - dn \\
&= \frac{d}{2}n(n-1) \quad .
\end{aligned}$$

We conclude that

$$S_{n,d}(p) = \Phi_{n,d}(p) \prod_{i=1}^n \Gamma(p+1+d(i-1))$$

with

$$(2.2) \quad \Phi_{n,d}(p) = \left(\sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \operatorname{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n (p+1+d(i+1))_{2\mu_i(\sigma)} \right) \quad .$$

a polynomial of degree $\leq \frac{d}{2}n(n-1)$. \square

We remark here that the bound $\deg(\Phi_{n,d}(p)) \leq \frac{d}{2}n(n-1)$ is certainly not optimal. To see this, note that the Pochhammer products

$$\prod_{i=1}^n (p+1+d(i+1))_{2\mu_i(\sigma)}$$

are all monic polynomials, and consequently when we sum over the arrangements in $(\mathfrak{S}_n)^d$, the $\text{sgn}(\sigma_1 \cdots \sigma_d)$ factors will lead to a complete cancellation of the leading terms. In fact, explicit computations of the right hand side of (2.1) reveal that at least for small n and d the actual degree of $\Phi_{n,d}(p)$ is $\frac{d}{4}n(n-1)$; that is, that the terms of degree $\frac{d}{2}n(n-1)$, $\frac{d}{2}n(n-1)-1$, \dots , $\frac{d}{4}n(n-1)+1$ all, quite remarkably, cancel. Unfortunately, we have yet to find a direct combinatorial argument as to why the first $\frac{d}{4}n(n-1)$ leading terms all cancel. However, in §3 we shall succeed not only in extending our results to the case of odd d , but also in obtaining a least upper bound on the degree of the polynomial factor $\Phi_{n,d}(p)$ for arbitrary positive integers d and n .

2.1. The case when $d = 2$. When $d = 2$ we can obtain a more succinct determinantal formula for the polynomial factor $\Phi_{n,d}(p)$. Starting with

$$S_{n,2}(p) = \sum_{\sigma \in \mathfrak{S}_n} \sum_{\rho \in \mathfrak{S}_n} \text{sgn}(\sigma) \text{sgn}(\rho) \prod_{i=1}^n \Gamma(2\sigma(i) + 2\rho(i) - 3 + p)$$

we can write

$$\prod_{i=1}^n \Gamma(2\sigma(i) + 2\rho(i) + p - 3) = \prod_{i=1}^n \Gamma(2i + 2\rho(\sigma^{-1}(i)) - 3 + p)$$

and then replace the sum over $\rho \in \mathfrak{S}_n$ with the sum over $\tau = \rho\sigma^{-1} \in \mathfrak{S}_n$. Noting that

$$\text{sgn}(\tau) = \text{sgn}(\rho) \text{sgn}(\sigma^{-1}) = \text{sgn}(\rho) \text{sgn}(\sigma) \quad ,$$

we obtain in this way

$$\begin{aligned} S_{n,2}(p) &= \sum_{\sigma \in \mathfrak{S}_n} \sum_{\tau \in \mathfrak{S}_n} \text{sgn}(\tau) \prod_{i=1}^n \Gamma(2i + 2\tau(i) - 3 + p) \\ &= n! \sum_{\tau \in \mathfrak{S}_n} \text{sgn}(\tau) \prod_{i=1}^n \Gamma(p + 2i - 1 + 2\tau(i) - 2) \\ &= n! \sum_{\tau \in \mathfrak{S}_n} \text{sgn}(\tau) \prod_{i=1}^n (p - 1 + 2i)_{2\tau(i)-2} \Gamma(p - 1 + 2i) \\ &= n! \det \left[(p - 1 + 2i)_{2j-2} \right]_{\substack{i=1, \dots, n \\ j=1, \dots, n}} \prod_{i=1}^n \Gamma(p - 1 + 2i) \end{aligned}$$

and so

$$\Phi_{n,2}(p) = n! \det \left[(p - 1 + 2i)_{2j-2} \right]_{\substack{i=1, \dots, n \\ j=1, \dots, n}} \quad .$$

Remark 2.1. This formula for $\Phi_{n,2}(p)$ appears as Remark (3.7) in [NOZ].

Remark 2.2. The above determinant formula of $\Phi_{n,2}(p)$ seems to predict a polynomial of total degree

$$\sum_{j=1}^n (2j - 2) = n(n + 1) - 2n = n(n - 1) = \frac{d}{2}n(n - 1)$$

in agreement with the upper bound stated in Prop. 2.2. But, as remarked above, explicit computations reveal that, somewhat miraculously, the first $(\deg \Phi_{n,d})/2$ leading terms all cancel and one has

$$\deg(\Phi_{n,2}(p)) = \frac{1}{2}n(n-1) = \frac{d}{4}n(n-1) \quad .$$

3. THE INTEGRAL FOR THE GENERAL CASE $d \in \mathbb{Z}_{>0}$

3.1. Symmetric polynomials and an integral formula of Macdonald. In order to keep the exposition of our results self-contained, we begin with a rapid review of the pertinent theory of symmetric functions. A standard reference for this material is, of course, [Mac2].

By a symmetric polynomial of n variables we mean a polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ invariant under the natural action of the symmetric group \mathfrak{S}_n :

$$p(x_1, \dots, x_n) = (\sigma \cdot p)(x_1, \dots, x_n) \equiv p(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \quad \forall \sigma \in \mathfrak{S}_n \quad .$$

The action of \mathfrak{S}_n preserves the subspaces of $\mathbb{C}[x_1, \dots, x_n]$ consisting of homogeneous polynomials of fixed total degree. We shall denote by $\Lambda_{(n)}^m$ the space of homogeneous symmetric polynomials in n variables of total degree m , so that

$$\Lambda_{(n)} \equiv \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n} = \bigoplus_{m=0}^{\infty} \Lambda_{(n)}^m \quad .$$

There are several fundamental bases for the subspaces $\Lambda_{(n)}^m$, each parameterized by partitions of length n and weight m . A partition λ of length n is simply a non-increasing list of non-negative integers; i.e., $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. The weight $|\lambda|$ of a partition λ is the sum of its parts; i.e., $|\lambda| = \sum_{i=1}^n \lambda_i$. There is a partial ordering of the set of partitions of weight w , called the *dominance partial ordering*, defined as follows

$$\mu \leq \lambda \quad \implies \quad |\lambda| = |\mu| \quad \text{and} \quad \sum_{j=1}^i (\lambda_j - \mu_j) \geq 0 \quad , \quad \text{for } i = 1, 2, \dots, n \quad .$$

Let $\lambda = [\lambda_1, \dots, \lambda_n]$ be a partition, and let $\mathbf{x}^\lambda = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ be the corresponding monomial. The *monomial symmetric function* m_λ is the sum of all distinct monic monomials that can be obtained from \mathbf{x}^λ by permuting the x_i 's. Every homogeneous symmetric polynomial of degree m can be uniquely expressed as a linear combination of the m_λ with $|\lambda| = m$.

The *power sum* symmetric polynomials are defined as follows. For each $r \geq 1$, let

$$p_r(\mathbf{x}) = m_{(r)}(\mathbf{x}) = \sum_{i=1}^n x_i^r$$

and then for any partition λ , set

$$p_\lambda(\mathbf{x}) = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots p_{\lambda_n}(x) \quad .$$

The power sum symmetric functions $p_\lambda(x)$ with $|\lambda| = m$ provide another basis for the homogeneous symmetric polynomials of degree m . In what follows, the power sum symmetric functions are only used to define a particular inner product for the symmetric polynomials; i.e., an inner product will be defined by specifying matrix entries with respect to the basis of power sum symmetric polynomials.

The *Schur polynomials* provide yet another basis. These can be defined as follows. For any partition λ of length n ,

$$a_\lambda(\mathbf{x}) = \det \left(x_i^{\lambda_j} \right) .$$

The $a_\lambda(\mathbf{x})$ are obviously odd with respect to the action of the permutation group \mathfrak{S}_n ; i.e. $a_\lambda(\sigma(\mathbf{x})) = \text{sgn}(\sigma) a_\lambda(\mathbf{x})$ for all $\sigma \in \mathfrak{S}_n$. However, it turns out that $a_\lambda = 0$ for and $\lambda < \delta \equiv [n-1, n-2, \dots, 1, 0]$, and that, for every partition λ , $a_\delta(\mathbf{x})$ divides $a_{\lambda+\delta}(\mathbf{x})$. Indeed,

$$s_\lambda(\mathbf{x}) \equiv \frac{a_{\lambda+\delta}(\mathbf{x})}{a_\delta(\mathbf{x})}$$

is a symmetric polynomial of degree $|\lambda|$. The polynomials $s_\lambda(\mathbf{x})$ are the Schur symmetric polynomials. The Schur polynomials $\{s_\lambda \mid |\lambda| = m\}$ provide another fundamental basis for the symmetric polynomials that are homogeneous of degree m . A special case that will be important to us later on is

$$s_\delta(\mathbf{x}) = \prod_{1 \leq i < j \leq n} (x_i + x_j)$$

where again $\delta = [n-1, n-2, \dots, 1, 0]$. We note also the fact that $a_\delta(\mathbf{x})$ is just the Vandermonde determinant

$$a_\delta(\mathbf{x}) = \det \begin{bmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{n-1}^{n-1} & x_{n-1}^{n-2} & \cdots & x_{n-1} & 1 \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{bmatrix} = \prod_{1 \leq i < j \leq n} (x_i - x_j) = \Delta(\mathbf{x}) .$$

Jack's symmetric functions $P_\lambda^{(\alpha)}$ are symmetric functions indexed by partitions and depending rationally on a parameter α which interpolate between the Schur functions $s_\lambda(\mathbf{x})$, the monomial symmetric functions and two other bases associated with spherical symmetric polynomials on symmetric spaces. They are (uniquely) characterized by two properties

- (i) $P_\lambda^{(\alpha)}(\mathbf{x}) = m_\lambda(\mathbf{x}) + \sum_{\substack{\mu < \lambda \\ |\mu| = |\lambda|}} c_{\lambda, \mu} m_\mu(\mathbf{x})$; that is, the ‘‘leading term’’ of

$P_\lambda^{(\alpha)}$ is the monomial symmetric functions m_λ and the remaining terms involve only monomial symmetric functions m_μ for which the partition index μ is less than λ with respect to the dominance ordering.

- (ii) When one defines a scalar product on the vector space of homogeneous polynomials of degree m by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda, \mu} \alpha^{\ell(\lambda)} \prod_{r=1}^n \left(r^{m_\lambda(r)} \cdot m_\lambda(r)! \right)$$

where $m_\lambda(r)$ is the number of times the integer r appears in λ , then

$$\langle P_\lambda^{(\alpha)}, P_\mu^{(\alpha)} \rangle = 0 \quad \text{if } \mu \neq \lambda .$$

We note that this definition basically ensures that the $P_\lambda^{(\alpha)}(\mathbf{x})$ are constructible via a Gram-Schmidt process (although not quite straightforwardly, as the dominance ordering is only a partial ordering).

In fact, the Jack symmetric polynomials $P_\lambda^{(\alpha)}$ for the particular values $\alpha = 2, 1$, and $\frac{1}{2}$ coincide (up to normalizing constants) with the spherical polynomials for, respectively, $GL(n, \mathbb{R})/O(n)$, $GL(n, \mathbb{C})/U(n)$, $GL(n, \mathbb{H})/U(\mathbb{H})$. The Jack symmetric polynomials $P_\lambda^{(1)}$ also coincide with the Schur polynomials s_λ .

Next we recall the following well known property of Schur polynomials

$$s_\mu(\mathbf{x}) s_\nu(\mathbf{x}) = \sum_{\lambda \leq \mu + \nu} K_{\mu\nu}^\lambda s_\lambda$$

where the coefficients $K_{\mu\nu}^\lambda$ are determined by the Littlewood-Richardson rule (and are, in fact, interpretable as Clebsch-Gordan coefficients for $SL(n)$). Because of the triangular decomposition of the product of two Schur polynomials in terms of other Schur polynomials, and the triangular decomposition of a Jack symmetric polynomial (Schur polynomials in particular) in terms of the monomial symmetric functions we can infer that

$$(3.1) \quad (s_\delta(\mathbf{x}))^d = \sum_{\lambda \leq d\delta} c_\lambda^{(\alpha)} P_\lambda^{(\alpha)}(x)$$

for suitable coefficients $c_\lambda^{(\alpha)}$ (Note that

$$\text{span} \{s_\lambda \mid \lambda \leq \delta\} = \text{span} \{m_\lambda \mid \lambda \leq \delta\} = \text{span} \{P_\lambda^{(\alpha)} \mid \lambda \leq \delta\}$$

which follows immediately from condition (i) in the definition of the $P_\lambda^{(\alpha)}$ and the linear independence of the $P_\lambda^{(\alpha)}$ which, in turn, follows immediately from orthogonality property (ii) in the definition of the $P_\lambda^{(\alpha)}$.)

We now quote a specialization of a result of Macdonald ([Mac2], pg. 386) that is, in turn, derived by applying a particular change of variables applied to a formula due to Gross and Richards [GR], and Kadell [Kad].

Lemma 3.1. For $\text{Re}(d) > 0$, $\text{Re}(p) > -1$,

$$\int_{\mathcal{S}_n} P_\lambda^{(\frac{2}{d})}(\mathbf{x}) \prod_{i=1}^n (x_i)^p (\Delta(\mathbf{x}))^d d\mathbf{x} = \frac{1}{\Gamma(a+1)} v_\lambda(d) \prod_{i=1}^n \Gamma(\lambda_i + p + 1 + d(n-i+1)/2)$$

where a is given by

$$a = |\lambda| + n(p + 1 + d(n-1)/2)$$

(the degree of the homogeneity of the integrand plus the number of variables) and

$$(3.2) \quad v_\lambda(d) = \prod_{1 \leq i < j \leq n} \frac{\Gamma(\lambda_i - \lambda_j + d(j-i+1)/2)}{\Gamma(\lambda_i - \lambda_j + d(j-i)/2)} .$$

Now note that the second factor in the integrand on the right hand side of (1.5) can be written as

$$\begin{aligned} \left(\prod_{1 \leq i < j \leq n} (x_i^2 - x_j^2) \right)^d &= \left(\prod_{1 \leq i < j \leq n} (x_i + x_j) \right)^d \left(\prod_{1 \leq i < j \leq n} (x_i - x_j) \right)^d \\ &= (s_\delta(\mathbf{x}))^d (\Delta(\mathbf{x}))^d . \end{aligned}$$

And so

$$I_{n,d,p} = \int_{\mathcal{S}_n} \left(\prod_{i=1}^n x_i^p \right) (s_\delta(\mathbf{x}))^d (\Delta(\mathbf{x}))^d d\mathbf{x} .$$

We now employ the expansion (3.1) with $\alpha = \frac{d}{2}$ to obtain

$$\begin{aligned} I_{n,d,p} &= \int_{\mathcal{S}_n} \left(\prod_{i=1}^n x_i^p \right) \left(\sum_{\lambda \leq d\delta} c_\lambda^{(\frac{2}{d})} P_\lambda^{(\frac{2}{d})}(\mathbf{x}) \right) (\Delta(\mathbf{x}))^d dx \\ &= \sum_{\lambda \leq d\delta} c_\lambda^{(\frac{2}{d})} \int_{\mathcal{S}_n} \left(\prod_{i=1}^n x_i^p \right) P_\lambda^{(\frac{2}{d})}(\mathbf{x}) (a_\delta(\mathbf{x}))^d dx \\ &= \frac{1}{\Gamma(a+1)} \sum_{\lambda \leq d\delta} c_\lambda^{(\frac{2}{d})} v_\lambda \left(\frac{d}{2} \right) \prod_{i=1}^n \Gamma \left(\lambda_i + r + \frac{d}{2} (n-i) \right). \end{aligned}$$

Using the fact that for each λ in the sum $|\lambda| = d|\delta| = dn(n-1)/2$ we obtain

Lemma 3.2. Let $\delta = [n-1, \dots, 2, 1, 0]$ and let $\left\{ c_\lambda^{(\frac{2}{d})} \right\}_{\lambda \leq d\delta}$ be the coefficients of Jack symmetric functions $P_\lambda^{(\frac{2}{d})}$ corresponding to the expansion

$$(3.3) \quad \left(\prod_{1 \leq i < j \leq n} (x_i + x_j) \right)^d = \sum_{\lambda \leq d\delta} c_\lambda^{(\frac{2}{d})} P_\lambda^{(\frac{2}{d})}$$

Then

$$(3.4) \quad I_{n,d,p} = \frac{1}{\Gamma(n(p+1+d(n-1))+1)} \sum_{\lambda \leq d\delta} c_\lambda^{(\frac{2}{d})} v_\lambda \left(\frac{d}{2} \right) \prod_{i=1}^n \Gamma(\lambda_i + p + 1 + d(n-i)/2)$$

with $v_\lambda(\frac{d}{2})$ as in (3.2).

Remark 3.1. So long as the total degree of the integrand is not too large (e.g. $d|\delta| \leq 16$), one can readily employ one J. Stembridge's SF package [St] to determine the coefficients $c_\lambda^{(\frac{2}{d})}$. Formula (3.4) then provides perhaps the most expedient method of evaluating the integrals $I_{n,d,p}$. For example, in the case where $d = 1$, one finds

n	$s_\delta = \prod_{1 \leq i < j \leq n} (x_i + x_j)$
2	$P_{[1]}^{(2)}$
3	$P_{[2,1]}^{(2)} + \frac{1}{2} P_{[1,1,1]}^{(2)}$
4	$P_{[3,2,1]}^{(2)} + \frac{1}{2} P_{[3,1,1,1]}^{(2)} + \frac{1}{2} P_{[2,2,2]}^{(2)} + \frac{16}{21} P_{[2,2,1,1]}^{(2)} + \frac{7}{20} P_{[2,1,1,1,1]}^{(2)}$
5	$P_{[4,3,2,1]}^{(2)} + \frac{1}{2} P_{[4,3,1,1,1]}^{(2)} + \frac{1}{2} P_{[4,2,2,2]}^{(2)} + \frac{16}{21} P_{[4,2,2,1,1]}^{(2)} + \frac{7}{20} P_{[4,2,1,1,1,1]}^{(2)}$ $+ \frac{1}{2} P_{[3,3,3,1]}^{(2)} + \frac{16}{21} P_{[3,3,2,2]}^{(2)} + \frac{10}{9} P_{[3,3,2,1,1]}^{(2)} + \frac{16}{33} P_{[3,3,1,1,1,1]}^{(2)} + \frac{182}{245} P_{[3,2,2,2,1]}^{(2)}$ $+ \frac{293}{396} P_{[3,2,2,1,1,1]}^{(2)} + \frac{1}{4} P_{[2,2,2,2,2]}^{(2)} + \frac{16}{45} P_{[2,2,2,2,1,1]}^{(2)} + \frac{7}{50} P_{[2,2,2,1,1,1,1]}^{(2)}$

Inserting these expansions into formula (3.4) yields

$$\begin{aligned}
I_{2,1,p} &= \frac{2}{\sqrt{\pi}\Gamma(2p+5)}\Gamma(p+1)\Gamma\left(p+\frac{5}{2}\right) \\
I_{3,1,p} &= \frac{8p+23}{2\sqrt{\pi}\Gamma(3p+10)}\left(4p+\frac{23}{2}\right)\Gamma(p+1)\Gamma\left(p+\frac{5}{2}\right)\Gamma(p+3) \\
I_{4,1,p} &= \frac{1}{\pi}\frac{1}{\Gamma(4p+17)}\left(32p^2+\frac{2247}{10}p+\frac{3787}{10}\right)\Gamma(p+1)\Gamma\left(p+\frac{5}{2}\right)\Gamma(p+3)\Gamma\left(p+\frac{9}{2}\right) \\
I_{5,1,p} &= \frac{1}{\pi}\frac{1}{\Gamma(5p+37)}\left(384p^4+\frac{878047}{132}p^3+\frac{1007527879}{23760}p^2+\frac{465238651}{3960}p+\frac{2822536697}{23760}\right) \\
&\quad \Gamma(p+1)\Gamma\left(p+\frac{5}{2}\right)\Gamma(p+3)\Gamma\left(p+\frac{9}{2}\right)\Gamma(p+5)
\end{aligned}$$

We shall now derive from Lemma 3.2 the general formula of Theorem 2.

Lemma 3.3. *If $\lambda = [\lambda_1, \dots, \lambda_n]$ is a partition of weight $d|\delta|$ such that $\lambda \leq d\delta$, then*

$$\lambda_i \geq \mu_i \equiv \left\lfloor \frac{d(n-i)+1}{2} \right\rfloor .$$

Proof. Let \mathcal{P} be the set of partitions $\lambda = [\lambda_1, \dots, \lambda_n]$ satisfying the criteria

$$\begin{aligned}
\lambda_1 &\geq \lambda_2 \geq \dots \geq \lambda_n \geq 0 \\
\sum_{i=1}^n \lambda_i &= \frac{d}{2}n(n-1) \\
\sum_{j=1}^i \lambda_j &\leq \sum_{j=1}^i d(n-j) = ni - \frac{1}{2}i(i+1) = \frac{d}{2}i(2n-i-1) .
\end{aligned}$$

Since the total weight of such a λ is fixed, in order to minimize a particular part λ_i we need to arrange it so that the parts λ_j to the left of λ_i are as large as possible while the λ_j to the right of λ_i are also large as possible (otherwise we could shift some of λ_i 's weight to the right). In fact, if

$$\mu_i \equiv \min_{\lambda \in \mathcal{P}} \lambda_i$$

we'll need

$$\mu_i = \mu_{i+1} = \dots = \mu_n$$

to make the μ_j to the right as large as possible and

$$\mu_j = d(n-j) \quad , \quad j = 1, \dots, i-1$$

for the μ_j to the left to be as large as possible. But for such a minimizing configuration μ , we must also have

$$\begin{aligned}
\frac{d}{2}n(n-1) &= \sum_{j=1}^n \mu_j = \sum_{j=1}^{i-1} d(n-j) + (n-i+1)\mu_i \\
&= dn(i-1) - \frac{d}{2}i(i-1) + (n-i+1)\mu_i .
\end{aligned}$$

Solving this for μ_i yields

$$\mu_i = \frac{d}{2}(n-i) .$$

However, if d is odd then μ_i will be an integer only when $n - i$ is even. In such a case, the first integer larger than μ_i would be

$$\frac{d}{2}(n-i) + \frac{1}{2}.$$

Accounting for this circumstance, we can write

$$\mu_i = \min_{\lambda \in \mathcal{S}} \lambda_i = \left\lceil \frac{d(n-i) + 1}{2} \right\rceil,$$

noting that the added $\frac{1}{2}$ is innocuous in the cases when d is even or when d is odd and $n - i$ is even. \square

Theorem 2. Let $I_{n,d,p}$ and $a = n(p + 1 + d(n - 1))$.

- If d is even, then $I_{n,d,p}$ is of the form

$$(3.5) \quad I_{n,d,p} = \frac{1}{\Gamma(a+1)} \Phi(p) \prod_{j=0}^{n-1} \Gamma(p + dj + 1)$$

with $\Phi(p)$ a polynomial in p of degree $\leq \frac{d}{4}n(n-1)$.

- If d is odd,

$$(3.6) \quad I_{n,d,p} = \frac{1}{\Gamma(a+1)} \Phi(p) \prod_{j=0, \text{even}}^{n-1} \Gamma(p + dj + 1) \prod_{j=1, \text{odd}}^{n-1} \Gamma(p + dj + 3/2)$$

with $\Phi(p)$ a polynomial in p of degree $\leq \frac{d}{2}n(n-1) - \sum_{i=1}^n \left\lceil \frac{d(n-i)+1}{2} \right\rceil$.

Proof. Let $C_\lambda = c_\lambda^{(\frac{2}{d})} v_\lambda(\frac{d}{2})$. Applying Lemmas 3.1 and 3.2 we have

$$\begin{aligned} I_{n,d,p} &= \frac{1}{\Gamma(n(p+1+d(n-1))+1)} \sum_{\lambda \leq d\delta} C_\lambda \prod_{i=1}^n \Gamma\left(\lambda_i - \mu_i + \mu_i + p + 1 + \frac{d}{2}(n-i)\right) \\ &= \frac{1}{\Gamma(n(p+1+d(n-1))+1)} \\ &\quad \times \sum_{\lambda \leq d\delta} C_\lambda \prod_{i=1}^n \left(\mu_i + p + 1 + \frac{d}{2}(n-i)\right)_{\lambda_i - \mu_i} \Gamma\left(\mu_i + p + 1 + \frac{d}{2}(n-i)\right) \\ &= \frac{1}{\Gamma(n(p+1+d(n-1))+1)} \left(\sum_{\lambda \leq d\delta} C_\lambda \prod_{i=1}^n \left(\mu_i + p + 1 + \frac{d}{2}(n-i)\right)_{\lambda_i - \mu_i} \right) \\ &\quad \times \left(\prod_{i=1}^n \Gamma(\mu_i + p + 1 + d(n-i)/2) \right) \\ &= \frac{1}{\Gamma(a+1)} \Phi(p) \Gamma_{n,d,p} \end{aligned}$$

where

$$\Phi(p) = \sum_{\lambda \leq d\delta} C_\lambda \prod_{i=1}^n \left(\mu_i + p + 1 + \frac{d}{2}(n-i)\right)_{\lambda_i - \mu_i}.$$

and

$$\Gamma_{n,d,p} \equiv \prod_{i=1}^n \Gamma(\mu_i + p + 1 + d(n-i)/2)$$

Noting

$$\Gamma_{n,d,p} = \prod_{j=0}^{n-1} \Gamma(\mu_{n-j} + p + 1 + dj/2)$$

and

$$\mu_{n-j} = \left[\frac{d(n - (n-j)) + 1}{2} \right] = \begin{cases} dj/2 & \text{if } d \text{ or } j \text{ is even} \\ (dj+1)/2 & \text{if } d \text{ and } j \text{ are odd} \end{cases}$$

we have

$$\Gamma_{n,d,p} = \begin{cases} \prod_{j=0}^{n-1} \Gamma(p+1+ dj) & (d \text{ even}) \\ \prod_{j=0, \text{even}}^{n-1} \Gamma(p+1+ dj) \prod_{j=1, \text{even}}^{n-1} \Gamma(p+ dj + \frac{3}{2}) & (d \text{ odd}) \end{cases}$$

and so the gamma factors of $I_{n,d,p}$ are of the stated form.

We'll now obtain the stated upper bounds for the degree of the polynomial factor Φ by determining the total degree of the products of Pochhammer symbols (as λ ranges over the set of partitions $\leq d\delta$).

$$\begin{aligned} \deg \Phi &\leq \deg \prod_{i=1}^n \left(\mu_i + p + 1 + \frac{d}{2}(n-i) \right)_{\lambda_i - \mu_i} \\ &= \sum_{i=1}^n (\lambda_i - \mu_i) \\ &= \sum_{i=1}^n \lambda_i - \sum_{i=1}^n \mu_i \\ &= \frac{d}{2}n(n-1) - \sum_{i=1}^n \left[\frac{d(n-i)+1}{2} \right] . \end{aligned}$$

Now when d is even, we have $\left[\frac{d(n-i)+1}{2} \right] = \frac{d}{2}(n-i)$ and so

$$\begin{aligned} \sum_{i=1}^n (\lambda_i - \mu_i) &= \frac{d}{2}n(n-1) - \sum_{i=1}^n \frac{d(n-i)}{2} = \frac{d}{2}n(n-1) - \frac{d}{2}n^2 + \frac{d}{4}n(n+1) \\ &= \frac{d}{4}n(n-1) \quad (\text{if } d \in 2\mathbb{Z}) . \end{aligned}$$

□

Remark 3.2. Explicit calculations reveal that for $d = 1, 2, 4$ and $n = 2, 3, 4, 5$ that the stated upper bounds on the degree of $\Phi(r)$ are, in fact, realized (and thus far without exception). Thus, $\frac{d}{4}n(n-1)$ is indeed the least upper bound for general d and n . We thus have a noncombinatorial proof of the following interesting and purely combinatorial statement:

$$\begin{aligned} &\sum_{\sigma_1 \in \mathfrak{S}_n} \cdots \sum_{\sigma_d \in \mathfrak{S}_n} \text{sgn}(\sigma_1 \cdots \sigma_d) \prod_{i=1}^n \Gamma(r + 2\sigma_1(i) + \cdots + 2\sigma_d(i)) \\ &= \Phi(r) \prod_{i=1}^n \Gamma(r + d(n-i)) \quad (d \in 2\mathbb{Z}) \end{aligned}$$

where $\Phi(r)$ is a polynomial of degree $\leq \frac{d}{4}n(n-1)$. (Our point here being to highlight again the remarkable fact that after removing the “greatest common gamma

factors" from the terms on the left hand side, exactly half of the leading terms cancel.)

4. RECURSIVE FORMULAS

In this section, we demonstrate another evaluation tactic for Selberg-like integrals; one aimed at exploiting the interplay between the integrand and the region of integration \mathcal{S}_n . We begin by considering integrals of the form

$$(4.1) \quad J_{n,\kappa}(f) \equiv \int_{\mathcal{S}_n} f(\mathbf{x}) \left(\Delta^{(n)}(\mathbf{x}) \right)^\kappa d\mathbf{x}$$

where f is some homogeneous symmetric function and \mathcal{S}_n is the same simplex as in (1.6) and $\Delta^{(n)}(\mathbf{x})$ is the Vandermonde determinant in n -variables.

$$\Delta^{(n)}(\mathbf{x}) = \left(\prod_{1 \leq i < j \leq n} (x_i - x_j) \right) .$$

Shortly, we shall specialize to the (still general) case when $f = \Phi_\lambda$, $\{\Phi_\lambda\}$ being some standard basis for homogeneous symmetric polynomials in n variables indexed of partitions λ .

4.1. A Change of Variables Formula. In what follows, it clarifies matters to indicate by $\mathbf{x}_{(n)}$ elements of \mathbb{R}^n , and by $\mathbf{x}_{(i)}$, $1 \leq i \leq n$, the vector in \mathbb{R}^i obtained by dropping the last $n - i$ components of $\mathbf{x}_{(n)}$.

Consider the following change of variables

$$(4.2) \quad \begin{aligned} t_i &= \frac{x_i - x_n}{1 - nx_n} , & x_i &= (1 - t_n)t_i + \frac{1}{n}t_n , \\ t_n &= nx_n , & x_n &= \frac{1}{n}t_n . \end{aligned}$$

It is easy to check that Jacobian of this transformation is

$$\left(\frac{\partial \mathbf{x}}{\partial \mathbf{t}} \right) = \det \begin{bmatrix} 1 - t_n & 0 & \cdots & \frac{1}{n} \\ 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 - t_n & \frac{1}{n} \\ 0 & \cdots & 0 & \frac{1}{n} \end{bmatrix} = \frac{1}{n} (1 - t_n)^{n-1}$$

and the new region of integration is

$$\left\{ \mathbf{t} \in \mathbb{R}^n \mid t_1 \geq t_2 \geq \cdots \geq t_{n-1} \geq 0 , \quad \sum_{i=1}^{n-1} t_i \leq 1 , \quad 0 \leq t_n \leq 1 \right\} \approx \mathcal{S}_{n-1} \times [0, 1] .$$

We thus have

Lemma 4.1.

$$\int_{\mathcal{S}_n} f(\mathbf{x}_{(n)}) d\mathbf{x}_{(n)} = \int_0^1 \frac{1}{n} (1 - t_n)^{n-1} \left(\int_{\mathcal{S}_{n-1}} f(\mathbf{x}_{(n)}(\mathbf{t}_{(n)})) d\mathbf{t}_{(n-1)} \right) dt_n$$

where

$$\begin{aligned}\mathbf{x}_{(n)} &\equiv (x_1, \dots, x_n) \quad , \\ \mathbf{t}_{(n)} &\equiv (t_1, \dots, t_n) \quad , \\ x_i(\mathbf{t}_{(n)}) &= (1 - t_n) t_i + \frac{1}{n} t_n \quad , \\ x_n(\mathbf{t}_{(n)}) &= \frac{1}{n} t_n \quad .\end{aligned}$$

Remark 4.1. If one iterates the change of variables formula, then after $n - 1$ reductions, one arrives at

$$\begin{aligned}\int_{\mathcal{S}_n} f(\mathbf{x}_{(n)}) d\mathbf{x}_{(n)} &= \int_0^1 \frac{1}{n} (1 - t_n)^{n-1} \left(\int_0^1 \frac{1}{n-1} (1 - t_{n-1})^{n-2} \dots \right. \\ &\quad \left. \dots \left(\int_{\mathcal{S}_1} f(\mathbf{x}(\mathbf{t})) dt_1 \right) \dots \right) dt_{n-1} dt_n \\ &= \frac{1}{n!} \int_0^1 (1 - t_n)^{n-1} \left(\int_0^1 (1 - t_{n-1})^{n-2} \dots \right. \\ &\quad \left. \dots \left(\int_0^1 f(\mathbf{x}(\mathbf{t})) dt_1 \right) \dots \right) dt_{n-1} dt_n\end{aligned}$$

where

$$\begin{aligned}x_1(\mathbf{t}) &= (1 - t_n)(1 - t_{n-1}) \dots (1 - t_2) t_1 + \frac{1}{2} (1 - t_n) \dots (1 - t_3) t_2 + \dots + \frac{1}{n} t_n \quad , \\ x_2(\mathbf{t}) &= \frac{1}{2} (1 - t_n) \dots (1 - t_3) t_2 + \frac{1}{3} (1 - t_n) \dots (1 - t_3) t_3 + \dots + \frac{1}{n} t_n \quad , \\ &\vdots \\ x_{n-1}(\mathbf{t}) &= \frac{1}{n-1} (1 - t_n) t_{n-1} + \frac{1}{n} t_n \quad , \\ x_n(\mathbf{t}) &= \frac{1}{n} t_n \quad .\end{aligned}$$

If we adopt the convention that

$$\prod_{k=n+1}^n (1 - t_k) \equiv 1 \quad ,$$

the formula for $x_i(\mathbf{t})$ can be written a bit more succinctly as

$$x_i = \sum_{j=i}^n \frac{1}{j} \left(\prod_{k=j+1}^n (1 - t_k) \right) t_j \quad .$$

It should be now clear that, via this cumulative change of variables, every monomial $x_1^{m_1} \dots x_n^{m_n}$ can be expressed as a sum of terms of the form

$$c t_1^{a_1} (1 - t_1)^{b_1} \dots t_n^{a_n} (1 - t_n)^{b_n}$$

and so the integral of a homogeneous symmetric polynomial $f(\mathbf{x}_{(n)})$ over \mathcal{S}_n can be reduced to a sum of products of beta-integrals.

4.2. **A recursive formula for $J_{n,\kappa}(\Phi_\lambda)$.** Now let $\{\Phi_\lambda\}$ be some basis for the homogeneous symmetric polynomials in n variables and consider integrals of the form

$$(4.3) \quad J_{n,\kappa}(\Phi_\lambda) \equiv \int_{S_n} \Phi_\lambda(\mathbf{x}) \left(\Delta^{(n)}(\mathbf{x})\right)^\kappa d\mathbf{x} \quad .$$

We begin by noting that every symmetric basis function $\Phi_\lambda(\mathbf{x}_{(n)}) = \Phi_\lambda(x_1, \dots, x_n)$ of degree $|\lambda|$, will have a ‘‘symmetric Taylor expansion’’ of the form

$$(4.4) \quad \Phi_\lambda(\mathbf{x}_{(n)}) = \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} c_{\lambda\mu} \Phi_\mu(\mathbf{x}_{(n-1)}) x_n^i$$

for suitable coefficients $c_{\lambda\mu}$. We will refer to the $c_{\lambda\mu}$ as the ‘‘generalized Taylor coefficients’’ for Φ_λ . Similarly, we can define ‘‘generalized binomial coefficients’’ $\binom{\lambda}{\mu}$ for the Φ_λ by the formula

$$\Phi_\lambda(\mathbf{x}_{(n)} + t\mathbf{1}_{(n)}) = \sum_{\mu} \binom{\lambda}{\mu} \Phi_\mu(\mathbf{x}_{(n)}) t^{|\lambda|-|\mu|} \quad .$$

In terms of these gadgets,¹ we see that under the substitutions

$$x_i = (1 - t_n)t_i + \frac{1}{n}t_n = (1 - t_n) \left(t_i + \frac{t_n}{n(1 - t_n)} \right) \quad ,$$

$$x_n = \frac{1}{n}t_n \quad ,$$

the right hand side of (4.4) becomes

$$\begin{aligned} & \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} c_{\lambda\mu} \Phi_\mu(\mathbf{x}_{(n-1)}(\mathbf{t}_{(n)})) x_n^i(\mathbf{t}_{(n)}) \\ &= \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} c_{\lambda\mu} \Phi_\mu \left((1 - t_n) \left(\mathbf{t}_{(n-1)} + \left(\frac{t_n}{n(1 - t_n)} \right) \mathbf{1}_{(n-1)} \right) \right) \left(\frac{t_n}{n} \right)^i \\ &= \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} (1 - t_n)^{|\mu|} c_{\lambda\mu} \Phi_\mu \left(\mathbf{t}_{(n-1)} + \left(\frac{t_n}{n(1 - t_n)} \right) \mathbf{1}_{(n-1)} \right) \left(\frac{t_n}{n} \right)^i \\ &= \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} \sum_{\nu} (1 - t_n)^{|\mu|} c_{\lambda\mu} \binom{\mu}{\nu} \Phi_\nu(\mathbf{t}_{(n-1)}) \left(\frac{t_n}{n(1 - t_n)} \right)^{|\mu|-|\nu|} \left(\frac{t_n}{n} \right)^i \\ &= \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} \sum_{\nu} (n)^{|\nu|-n-i} (t_n)^{|\mu|-|\nu|+i} (1 - t_n)^{|\nu|} c_{\lambda\mu} \binom{\mu}{\nu} \Phi_\nu(\mathbf{t}_{(n-1)}) \end{aligned}$$

¹These ‘‘gadgets’’ are in fact interesting mathematical objects in their own right (see, for example, [VK] §2.5.1

and

$$\begin{aligned}
(\Delta(\mathbf{x}_{(n)}))^\kappa &= \prod_{1 \leq i < j \leq n} (x_i - x_j)^\kappa \\
&= \prod_{1 \leq i \leq n-1} (x_i - x_n)^\kappa \prod_{1 \leq i < j \leq n-1} (x_i - x_j)^\kappa \\
&= \prod_{1 \leq i \leq n-1} ((1-t_n)t_i)^\kappa \prod_{1 \leq i < j \leq n-1} ((1-t_n)(t_i - t_j))^\kappa \\
&= (1-t_n)^{\kappa(n-1 + \frac{1}{2}(n-1)(n-2))} \left(\prod_{1 \leq i \leq n-1} t_i \right)^\kappa \left(\prod_{1 \leq i < j \leq n-1} (t_i - t_j) \right)^\kappa \\
&= (1-t_n)^{\kappa n(n-1)/2} \left(\prod_{1 \leq i \leq n-1} t_i \right)^\kappa \left(\prod_{1 \leq i < j \leq n-1} (t_i - t_j) \right)^\kappa.
\end{aligned}$$

Applying Lemma 4.1, we thus obtain

$$\begin{aligned}
J_{n,\kappa}(\Phi_\lambda) &= \int_{S_n} \Phi_\lambda(\mathbf{x}_{(n)}) \left(\prod_{1 \leq i < j \leq n} (x_i - x_j) \right)^\kappa d\mathbf{x}_{(n)} \\
&= \int_0^1 \frac{1}{n} (1-t_n)^{n-1} \\
&\quad \times \left(\int_{S_{n-1}} \Phi(\mathbf{x}_{(n)}(\mathbf{t}_{(n)})) \prod_{1 \leq i < j \leq n} (x_i(\mathbf{t}_{(n)}) - x_j(\mathbf{t}_{(n)}))^t d\mathbf{t}_{(n-1)} \right) dt_n \\
&= \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} \sum_{\nu} (n)^{|\nu|-n-i-1} c_{\lambda\mu} \left(\begin{matrix} \mu \\ \nu \end{matrix} \right) \\
&\quad \times \left(\int_0^1 (t_n)^{|\mu|-|\nu|+i} (1-t_n)^{n-1 + \frac{\kappa}{2}n(n-1) + |\nu|} dt_n \right) \\
&\quad \times \int_{S_{n-1}} \Phi_\nu(\mathbf{t}_{(n-1)}) \prod_{1 \leq i < j \leq n-1} (t_i - t_j)^\kappa d\mathbf{t}_{(n-1)}.
\end{aligned}$$

Employing Euler's formula for the beta function

$$B(r, s) \equiv \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \int_0^1 t^{r-1} (1-t)^{s-1} dt$$

we thus obtain

Theorem 3.

$$\begin{aligned}
J_{n,\kappa}(\Phi_\lambda) &= \sum_{i=0}^{|\lambda|} \sum_{|\mu|=|\lambda|-i} \sum_{\nu} (n)^{|\nu|-n-i-1} c_{\lambda\mu} \left(\begin{matrix} \mu \\ \nu \end{matrix} \right) \\
&\quad \times B\left(|\lambda| + |\nu| + 1, n - 1 + \frac{\kappa}{2}n(n-1) + |\nu|\right) J_{n-1,\kappa}(\Phi_\nu).
\end{aligned}$$

REFERENCES

- [As] R. Askey, *Some basic hypergeometric extensions of integrals of Selberg and Andrews*, SIAM J. Math. Anal., **11** (1981), 938-951.

- [Ao] K. Aomoto, *Jacobi polynomials associated with Selberg integrals*, SIAM J. Math. Anal., **18** (1987), 545-549.
- [B] B. Binegar, *On a class of multiplicity-free $K_{\mathbb{C}}$ -orbits*, to appear in J. Math. Kyoto Univ. (preprint available at [arXiv:math.RT/0608167](https://arxiv.org/abs/math.RT/0608167), <http://arxiv.org/abs/math.RT/0608167>).
- [FK] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.
- [Fo] P. Forrester, *Integration formulas and exact calculations in the Calogero-Sutherland model*, Modern Phys. Letters B, **9** (1995), 359-371.
- [GR] K.I. Gross and D. St.P. Richards, *Special functions of matrix argument. I. Algebraic induction, zonal polynomials and hypergeometric functions*, Trans. Amer. Math. Soc. **301** (1987), 781-811.
- [Kad] K. Kadell, *The Selberg-Jack symmetric functions*, Adv. in Math., **130** (1997), 33-102.
- [Kan] J. Kaneko, *Selberg integrals and hypergeometric functions associated with Jack polynomials*, SIAM J. Math. Anal., **24** (1993), 1086-1110.
- [KO] S. Kato and H. Ochiai, *The degrees of orbits of the multiplicity free actions*, in “Nilpotent Orbits, Associated Cycles and Whittaker Models for Highest Weight Representations”, Astérisque **273**, Société Math. France (2001).
- [Kor] A. Korányi, *The volume of symmetric domains, the Koecher gamma function and an integral of Selberg*, Studia Scien. Math. Hungarica, **17** (1982), 129-133.
- [Mac1] I.G. Macdonald, *Some conjectures about root systems*, SIAM J. Math. Anal., **13** (1982), 988-1007.
- [Mac2] I.G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1995.
- [Meh] M.L. Mehta, *Random Matrices*, Academic Press, Boston, MA, 1991.
- [NOTYK] K. Nishiyama, H. Ochai, K. Taniguchi, H. Yamashita and S. Kato, *Nilpotent Orbits, Associated Cycles and Whittaker Models for Highest Weight Representations*, Asterisque, **273**, Soc. Math de France, Paris (2001).
- [NO] K. Nishiyama and H. Ochiai, *The Bernstein degree of singular unitary highest weight representations of the metaplectic group*, Proc. J. Acad., **75** (1999), 9-11.
- [NOZ] K. Nishiyama, H. Ochiai, and C. Zhu, *Theta liftings of nilpotent orbits for symmetric pairs* (preprint)
- [Ri] D. St. P. Richards, *Analogues and extensions of Selberg’s integral*, in “Proceedings IMA Workshop on q -series and Partitions, March 1988”
- [Se] A. Selberg, *Bemerkninger om et multipelt integral*, Norsk. Mat. Tidsskr, 26] (1944), 71-78.
- [St] J. Stembridge, *SF: a Maple Package for Symmetric Functions*, available at <http://www.math.lsa.umich.edu/~jrs/maple.html>.
- [Ve] J. Verbaarschot, *Spectral sum rules and Selberg’s integral formula*, Phys. Lett. B, **329** (1994), 351-357.
- [VK] N.J. Vilenkin and A.U. Klimyk, *Representations of Lie Groups and Special Functions*, Recent Advances, Kluwer Acad. Pub., Dordrecht (1995).

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