Shared Orbits

OSU Lie Groups Seminar

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1. Introduction/Motivation

For some time now I've been trying to get a handle on a means of specifying the annihilators of unipotent representations. The basic prototype of the what I have been looking for is Devra Garfinkle's description of the generators of the annihilators of the minimal representations.

Let G be a simply-connected complex Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{N}$ be the cone of nilpotent elements of $\mathfrak{g}$. It is well known that $\mathcal{N}$ consists of only finitely many G orbits, and that there is a unique nilpotent orbit $\mathcal{O}_{\min}$ of minimal dimension. In fact, one can explicitly describe the polynomial generators of the radical ideal in $S(\mathfrak{g})$ corresponding to $\overline{\mathcal{O}}_{\min}$. Let F_μ denote the irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight μ, and let λ be the highest root of $\mathfrak{g}$. Then the space $S^2(\mathfrak{g})$ of homogeneous polynomials of degree two decomposes under the adjoint action of G as

$$S^2(\mathfrak{g}) \approx F_{2\lambda} \oplus F_0 \oplus F_{\mu_1} \oplus \cdots \oplus F_{\mu_k}$$

(the point being that the summands corresponding to the trivial representation and the representation with highest weight 2λ always appear). Then

$$I_{\mathcal{O}_{\min}} = S(g)\left(F_0 \oplus F_{\mu_1} \oplus \cdots \oplus F_{\mu_k}\right)$$

Let $\pi_{\min}$ be the unipotent representation of G attached to the minimal orbit. Then it is known that $\pi_{\min}$ is unitary and the annihilator of $\pi_{\min}$ (the so-called Joseph ideal) is generated by

$$Sym\left((F_0 - \lambda_{\min}) \oplus F_{\mu_1} \oplus \cdots \oplus F_{\mu_k}\right)$$

where $Sym : S(\mathfrak{g}) \to U(\mathfrak{g})$ is the symmeterizer map. (This is essentially Garfinkle's result.)

In trying to generalize this picture for other nilpotent orbits and their corresponding unipotent representations one eventually has to confront a problematic example discovered by Joseph: the 8-dimensional nilpotent orbit $\mathcal{O}_8$ of G_2. It turns out that there are a number of peculiarities associated with this orbit:

- There are two completely prime primitive ideals attached to this orbit (only one of which corresponds to a unitary representation).
- One of these ideals (curiously enough, the one annihilating a unitary representation of G) has the property that $Gr(J)$ is not prime (because $Gr(J) \neq \sqrt{Gr(J)}$).
- The closure of $\mathcal{O}_8$ is not a normal variety, but $\mathcal{O}_8$ embeds densely in the minimal orbit of $B_3 \approx \mathfrak{so}(7,\mathbb{C})$ which is a normal variety, and the unitary representation attached to $\mathcal{O}_8$ can be realized as the (irreducible!) restriction of the minimal representation of B_3 to G_2.

It turns out also that the 10-dimensional nilpotent orbit of G_2 embeds densely in the minimal orbit of a larger simple group (in this case $D_4 \approx \mathfrak{so}(8,\mathbb{C})$). Moreover, the minimal representation of D_4 upon restriction to G_2 reveals a sort of dual pair theta correspondence: g_2 is the subalgebra of fixed points for the outer automorphisms corresponding to the S_3 symmetry ("triality") of the Dynkin diagram of D_4 and

$$\pi_{\min,\mathfrak{so}(8)}\big|_{\mathfrak{g}_2 \times S_3} = \bigoplus_{\sigma \in \widehat{S_3}} V_\sigma \otimes E_\sigma$$

Here the V_σ comprise the special unipotent representations with infinitesimal character $\omega_1 + \omega_2$ and E_σ is the irreducible S_3-module corresponding to σ.

This situation raised a bunch of questions for me; the main one being how prevalent is this sort of phenomenon; which in turn had two parts:

- When does one have dense embeddings
$$\mathfrak{g} \supset \mathcal{O} \hookrightarrow \mathcal{O}' \subset \mathfrak{g}'$$
of one nilpotent orbit into a nilpotent orbit of a larger Lie algebra. (I should remark that I was interested in this because the annihilators of a non-minimal unipotent representation of the smaller group might be most easily revealed by studying the embedding of $U(\mathfrak{g})$ into the Joseph ideal of $U(\mathfrak{g}')$.
- Is there always an accompanying dual pair phenomenon that could be used to provide realizations of non-minimal unipotent representations?

Unfortunately, the answers to both questions have already been attained. R. Brylinski and Kostant answered the first, and Jing-Song Huang answered the second (at least for simple complex groups). But it's such a pretty story, I thought it be a nice topic for a Lie groups seminar.

2. Normality and Shared Orbits (Brylinski/Kostant)

The following is a outline of how Brylinski and Kostant classified of dense embeddings of the form

$$\mathfrak{g} \supset \mathcal{O} \hookrightarrow \mathcal{O}' \subset \mathfrak{g}' \tag{2.1}$$

with $\mathfrak{g}$ a subalgebra of $\mathfrak{g}'$. Although they considered the general case when $\mathfrak{g}$ is a semisimple complex Lie algebra, in this outline I'll mostly stick to the case when $\mathfrak{g}$ is simple (because in the end, it turns out that if $\mathfrak{g}$ is simple, then a dense embedding like (2.1) is only possible when $\mathfrak{g}'$ is also simple - and that if $\mathfrak{g}$ is semisimple, then the allowed embeddings are effectively enumerated simple factor by simple factor).

2.1. Normal Closures of Orbit Coverings. Let G be a simply connected semisimple complex Lie group with Lie algebra $\mathfrak{g}$. Let $\mathcal{O}$ be the adjoint orbit of a nilpotent element $e \in \mathfrak{g}$, and let

$$\nu : M \to \mathcal{O}$$

be a G-homogeneous covering and choose $\varepsilon \in M$ such that $\nu(\varepsilon) = e$. Then

$$G_o^e \subset G^\varepsilon \subset G^e$$

and

$$\begin{aligned} M &\approx G/G^\varepsilon \\ \pi_1(M) &\approx G^\varepsilon/G_o^e \end{aligned}$$

Let $Q = Q(M)$ denote the group of all maps $\alpha : M \to M$ that commute with the action of G. Then

$$M \approx N^\varepsilon/G^\varepsilon$$

where N^ε is the normalizer of G^ε in G. $G \times Q$ acts rationally by algebra automorphisms on $R = R(M)$, the algebra of regular functions on M.

Let $(\ , \ )$ be the Killing form on $\mathfrak{g}$ (or any fixed $\mathfrak{g}$-invariant nonsingular symmetric bilinear form on $\mathfrak{g}$ that is negative definite on some compact form of $\mathfrak{g}$). For $x \in \mathfrak{g}$ define $\phi^x \in R$ by

$$\phi^x(p) = (\nu(p), x) \qquad , \qquad \forall\, p \in M$$

and set

$$R[\mathfrak{g}] = R(M)[\mathfrak{g}] = \text{linear span of the functions } \phi^x, \text{ where } x \in \mathfrak{g}$$

Clearly, as vector spaces $R[\mathfrak{g}] \approx \mathfrak{g}$. Furthermore, if $\overline{\mathcal{O}}$ is the closure of $\mathcal{O}$, then the subalgebra $S \subset R$ generated by $R[\mathfrak{g}]$ can be identified with $R(\overline{\mathcal{O}})$.

Lemma 0.1. *If Z is a normal algebraic variety, then the ring $R(Z)$ is integrally closed in the field $K(Z)$ of rational functions on Z.*

PROPOSITION 0.2.
- *There exists a unique affine variety X containing M as a Zariski open subset such that all regular functions on M extend to X.*[1]
- *The ring $R(M) = R(X) = R$ is a finitely generated $\mathbb{C}$-algebra and*

$$X = Spec(R) \quad .$$

- *The commuting actions of G and Q on M extend uniquely to commuting algebraic actions of G and M on X.*
- *The covering map ν extends uniquely to a finite surjective G-equivariant morphism*

$$\overline{\nu} : X \to \overline{\mathcal{O}}$$

- *X is a normal variety and in fact X is the normalization of $\overline{\mathcal{O}}$ in the function field of M.*
- *G has finitely many orbits on X and each is even dimensional*
- *M is the unique Zariski dense open orbit of G on X and its boundary has codimension at least 2.*

We call X the **normal closure of** M.

2.2. The Right Scaling Action of $\mathbb{C}^*$ on X and the graded Poisson Structure on $R(X)$. Recall that $\mathcal{O} = G \cdot e$, $M = G \cdot \varepsilon$. The Jacobson-Morosov Theorem says that there are $h, f \in \mathfrak{g}$ such that $\{e, f, h\}$ span an $\mathfrak{sl}(2, \mathbb{C})$ subalgebra of $\mathfrak{g}$, with

$$[h, e] = 2e \qquad , \qquad [h, f] = -2f \qquad , \qquad [e, f] = h$$

Then $\exp(\mathbb{C}h) \subset N^{\varepsilon} \subset G$, and so defines a subgroup C of $Q = N^{\varepsilon}/G^{\varepsilon}$, the group of all maps $M \to M$ that commute with the action of G), and hence C acts on X. It turns out that the vector field corresponding to the infinitesimal action of C on $\mathcal{O}$ is just two times the restriction of the Euler operator on $S(\mathfrak{g})$ to $\mathcal{O}$ (the Euler operator happens to be a G-invariant differential operator tangent to every orbit).

LEMMA 0.3. *The action of C on X lifts the square of the Euler action on $\overline{\mathcal{O}}$ so that $\overline{\nu}\left(\overline{e^{th}}x\right) = e^{2t}\overline{\nu}(x)$ for all $t \in \mathbb{C}$.*

Write $k \in \mathbb{Z}$, let

$$R[k] = \left\{\phi \in R \mid \left(e^{th}\phi\right)(x) = e^{tk}\phi(x)\right\}$$

PROPOSITION 0.4.
- *The action of C on $R = R(X)$ is completely reducible, and in fact,*

$$R = \bigoplus_{k=0}^{\infty} R[k]$$

is a G-invariant algebra grading.
- *Each $R[k]$ is a finite-dimensional G-stable subspace.*
- *$R[0] = \mathbb{C} \cdot 1$, where 1 is the constant function on X.*
- *There is a unique point $o \in X$ such that $\overline{\nu}(o) = 0 \in \overline{\mathcal{O}}$. This o is the unique G-fixed point of x and also the unique C-fixed point of x.*
- *Let $\mathfrak{m} \subset R$, be the maximal ideal at o. Then*

$$\mathfrak{m} = \bigoplus_{k=1}^{\infty} R[k]$$

and also $\mathfrak{m}$ is the sum in R of all non-trivial G-modules.
- *If the degree of the cover v is odd (e.g., if $M = \mathcal{O}$), then $R[k] = 0$ if k is odd.*

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[1] It seems to me that the condition that the regular functions on M be extendable to regular functions of the variety in which it embeds densely is really the crux of the utility such an embedding, and so this condition is a very natural hypothesis. It is remarkable, therefore, that from this requirement alone, we get a unique candidate for X.

Recall that an adjoint orbit admits a canonical G-invariant symplectic form $\omega_{\mathcal{O}}$. But then the pullback of $\omega = \omega_M = \nu^* (\omega_{\mathcal{O}})$ defines a symplectic form on M. Each $\phi \in R = R[M] = R[X]$ then defines a a Hamiltonian vector field ξ_ϕ on M by

$$\xi_\phi \rfloor \omega = d\phi$$

Then R is a Poisson algebra with Poisson bracket given by

$$\{\phi, \psi\} = \xi_\phi \psi = \omega (d\phi, d\psi)$$

Proposition 0.5. *The Poisson bracket established above make M a Hamiltonian G-space. Moreover,*

- *The right scaling action on M scales ω so that*
$$\{R[k], R[l]\} \subset R[k+l-2]$$
- *The map $\rho : \mathfrak{g} \to R^2$, $\rho(x) = \phi^x$ is a Lie algebra homomorphism.*
- *$R[2]+R[1]+R[0]$ is the unique maximal finite dimensional subalgebra of R containing $R[\mathfrak{g}] = \rho(\mathfrak{g})$*
- *If $\mathfrak{g}$ is semisimple, then $R[2]$ is semisimple. If $\mathfrak{g}$ is simple, then $R[2]$ is simple.*
- *If $R[1] \neq 0$, then the Poisson bracket gives $R[1] + R[0]$ the structure of a Heisenberg algebra and the bracket operation of $R[2]$ on $R[1]$ defines a Lie algebra surjection*
$$\delta : R[2] \to \mathfrak{sp}(2n, \mathbb{C}) \qquad , \qquad 2n = \dim R[1]$$

Now it is important to note that at this point we have concrete realization of the algebraic variety X (other than as $Spec(R(M))$). What makes everything about $R[2]$ computable is the idea of Algebraic Frobenius Reciprocity developed in Kostant's famous paper on rings of polynomials over $\mathfrak{g}$.

Fact 0.6 (Algebraic Frobenius Reciprocity). *For every G-module V there is a G-linear isomorphism*

$$t : V^{G^\varepsilon} \to Hom_G (V^*, R(G/G^\varepsilon))$$

defined by

$$t(v)(\gamma)(g \cdot \varepsilon) = \langle g \cdot v, \gamma \rangle \qquad , \qquad \forall\ g \in G \quad , \quad \forall\ v \in V^{G^\varepsilon} \quad , \quad \gamma \in V^*$$

Lemma 0.7. *For every G-module V and $k \in \mathbb{Z}$, t defines by restriction of $V^{G^\varepsilon}[k]$ a linear isomorphism*

$$t_k : V^{G^\varepsilon}[k] \to Hom(V^*, R[k])$$

In other words, we can explicitly enumerate what $\mathfrak{g}$-types appear in $R[k]$.

Example 0.8. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and let M be the simply-connected 3-fold cover of the principle nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}$. Let $V \approx \mathbb{C}^3$ be the standard representation, so that h has eigenvalues -2, 0, and 2 on V. One finds that the spaces

$$V^{\mathfrak{g}^\varepsilon}[2] \qquad , \qquad \left(\wedge^2 V\right)^{\mathfrak{g}^\varepsilon}[2] \qquad , \qquad \text{and} \qquad (\mathfrak{g})^{\mathfrak{g}^\varepsilon}[2]$$

are all 1-dimensional, and so the simple modules $\mathbb{C}^3$, $\wedge^2\mathbb{C}^3$, and $\mathfrak{g}$ all occur exactly once in $R[2]$. Furthermore, if $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are submodules of $R[2]$ carrying, respectively, $\mathbb{C}^3$ and $\wedge^2\mathbb{C}^3$. Then $\mathfrak{r} = \mathfrak{g} + \mathfrak{q}_1 + \mathfrak{q}_2$ is a 14-dimensional algebra semisimple algebra of rank 2. This already limits $\mathfrak{r}$ to $\mathfrak{g}_2$.