Characteristic Cycles for a Class of Small Unitary Representations, II

Lie Groups Seminar

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1. Review

2. Some Commutative Algebra

Here we review in just a wee bit more detail the commutative algebra that underlies notion of a characteristic cycle.

Definition 2.1. *A commutative ring R is* ***Noetherian*** *if every chain of ideals in R*

$$I_0 \subset I_1 \subset I_2 \subset \cdots$$

terminates after a finite number of steps (i.e., there is an interger k such that $I_s = I_k$ if $s \geq k$).

Remark 2.2. Polynomial rings $R[X_1, \dots, X_n]$ are Noetherian if R is. In particular, $S(\mathfrak{p}) = \mathbb{C}[\mathfrak{p}^*]$ is Noetherian.

Theorem 2.3. *If R is a Noetherian ring, and M is a finitely generated R-module, then any increasing filtration of M*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots$$

must terminate after a finite number of steps.

Definition 2.4. *An ideal P in a commutative ring R is said to be* ***prime*** *if*

(i) $P \neq R$
(ii) $x, y \notin P \quad \Rightarrow \quad xy \notin P$ *for all* $x, y \in R$.

Definition 2.5. *$Spec(R)$ is the set of all prime ideals of R.*

Remark 2.6. In the setting of algebraic geometry where R is a ring of polynomials over $\mathbb{C}$, prime ideals of R are in a one-to-one correspondences with irreducible affine varieties.

Definition 2.7. *Let M be a module for a commutative ring R. The set $Ass(M)$ of associated primes is the set of prime ideals P in R such P is the annihilator of some $m \in M$. That is to say,*

$$Ass(M) = \{P \in Spec(R) \mid P = Ann(m) \quad \text{for some } m \in M\}$$

Remark 2.8. If $P \in Ass(M)$ then necessarily there is a submodule of M isomorphic to R/P; viz;

$$R \cdot m \simeq R/P \qquad \text{if} \qquad P = Ann(m)$$

Theorem 2.9. *Let R be a Noetherian ring and $M \neq 0$ a finitely generated R-module. Then there exists a finite filtration*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

of M by submodules M_i such that for each i we have

$$M_i/M_{i-1} \simeq R/P_i$$

with $P_i \in Spec(R)$.

Proof. Choose any $P_1 \in Ass\,(M)$, then there exists a submodule M_1 of M isomorphic to R/P_1. Now choose any $P_2 \in Ass\,(M/M_1)$. Then there exists a submodule of $M_2 \subset M$ such that $M_2/M_1 \simeq R/P_2$. Continuing in this fashion we build an ascending chain of submodules of M. Since R is Noetherian and M is finitely generated, this ascending chain of submodules must eventually arrive at M.

Lemma 2.10. *Let R be a commutative Noetherian ring and let M be a finitely generated R-module. Choose a filtration*

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$$

of M by submodules M_i such that for each $i = 1, \ldots, n$, $M_i/M_{i-1} \simeq R/P_i$, with $P_i \in Spec\,(M)$. Let $S = \{P_i \mid i = 1, \ldots, n\}$, and let $S_{\min} = \{P \in S \mid P \subsetneq P' \quad \forall\, P' \in S\}$. Then, for $P \in S_{\min}$, the integers

$$m\,(M, P) = \#\,\{P' \in S \mid P = P'\}$$

is independent of the choice of filtration.

Remark 2.11. This lemma is not even a lemma in Vogan's paper; yet it is the basis for the definition of Vogan's definition of the characteristic cycle. As such it's a bit of a shame that he only gives some rough indications as to how one might prove it. Here's the idea though.

- Consider a particular submodule $M_i = R/P_i$ occuring in a filtration $\mathcal{F}$ of M. If P_i is not a maximal ideal, then it'll sit inside a chain of prime ideals terminating with a maximal ideal $P_{i,k}$

  $$P_i = P_{i,1} \subset P_{i,2} \subset \cdots \subset P_{i,k}$$

  And we'll have a corresponding filtration of M_i by submodules

  $$R/P_{i,k} \subset \cdots \subset R/P_{i,2} \subset R/P_{i,1} = M_i$$

- Lemma: If $\mathcal{F}'$ is a refinement of $\mathcal{F}$ obtained by inserting chains $M_{i,1} \subset \cdots \subset M_{i,k}$ in place of the original submodules $M_i = R/P_i$, then the chains must be of the form $R/P_{i,k} \subset \cdots \subset R/P_{i,2} \subset R/P_{i,1}$, with $P_i \subset P_{i,j}$ for $j = 2, \ldots, k$.
- Lemma (?) A common refinement exists for any pair of filtrations of M.[1]
- Notice that the process of refinement never introduces any new minimal primes to filtration. Therefore, the number of minimal primes in the common refinement of two filtrations must coincide with the number of minimal primes of each of the original filtrations.

3. Computing Multiplicities

In [V], the following proposition is given.

Proposition 3.1. *Suppose P is a prime ideal in a commutative Noetherian ring R, and M is a finitely generated R-module annihilated by P. There there is a element $f \in R - P$, with the property that M_f is a free $(R/P)_f$-module. Morever, $\mathrm{rank}\,(M_f)$ is the multiplicity of P in characteristic cycle of M.*

Corollary 3.2. *In the setting of the preceding proposition, suppose $\mathfrak{m}$ is any maximal ideal containing P but not containing f. Then*

$$m\,(P, M) = \dim_{R/\mathfrak{m}}\,(M/\mathfrak{m}M)$$

[1] That a common refinement exists is not at all obvious to me. What *might* be plausible is that any maximal refinement of a filtration of M has the same number of steps and up to changes of orderings and up to equivalences of the sort

$$P \sim P' \iff R/P \simeq R/P'$$

the same equivalence classes of prime ideals occur with the same multiplicity. But ultimately, IMO, this is business of a common refinement might be red herring. Indeed, it seems to me that the crux of the definition of multiplicity lies in the existence of a common minimal collapse of fitrations, rather than common refinements.

This formula however is not particularly amenable to direct calculation.

Here is an alternative procedure. Let $M = gr(X)$ be the $(S(\mathfrak{p}), K)$-module obtained from a good filtration of an irreducible admissible Harish-Chandra module and let

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M \tag{3.1}$$

be a finite filtration of M such that each $M_i/M_{i-1} \simeq S(\mathfrak{p})/P_i,\ P_i \in Spec(P_i)$. Now each prime ideal corresponds to P_i an irreducible affine variety $\mathcal{V}_i \subset \mathfrak{p}$. So (3.1) is a the filtration of $gr(X)$ by submodules such that $M_i/M_{i-1} \simeq S(\mathfrak{p})R/P_i \simeq R[\mathcal{V}_i]$ the ring of regular functions on $\mathcal{V}_i$.

Next, notice that if the filtration is (3.1) is $K_{\mathbb{C}}$-invariant, in which case the corresponding varieties $\mathcal{V}_i \subset \mathfrak{p}$ are the Zariski closures of $K_{\mathbb{C}}$-orbits, and when we filter M by K-types, we'll have

$$\dim M_\lambda = \sum_{i=1}^{n} \dim (S(\mathfrak{p})/P_i)_\lambda * \dim V_\lambda = \sum_i \dim R[\mathcal{V}_i]_\lambda$$

This formula shows that the Hilbert polynomial for M (using the filtration by $|\lambda|$) will just be the sum of the Hilbert polynomials for the $R[\mathcal{V}_i]$. But

$$\sum_{|\lambda|\leq k} \dim R[V_i]_\lambda \sim t^{\dim(\mathcal{V}_i)}$$

(Actually, this fact is one way of defining the dimension of an affine variety). So if d is the maximal dimension of the $\mathcal{V}_i$, only those V_i of dimension d will contribute to the leading term of the Hilbert polynomial for M. Thus,

$$p_M(t) = \sum_{\dim \mathcal{V}_i = d} p_{R[\mathcal{V}_i]}(t) + \text{ lower order terms}$$

Now

- The $\mathcal{V}_i$ for which $\dim \mathcal{V}_i = d$ will precisely the irreducible components of the associated variety $AV(\pi)$ that have dimension $\frac{1}{2}GK\dim(\pi)$.
- These varieties of maximal dimension in $AV(\pi)$ will correspond to minimal prime ideals occurring in the characteristic cycle of π, and the number of times such an minimal prime ideal occurs is by definition its multiplicity in the characteristic cycle.
- Identifying the minimal primes that occur in the characteristic cycle, with the corresponding irreducible affine varieties $\mathcal{V}_i \subset \mathfrak{p}$, which in turn are identified as with the closures of certain $K_{\mathbb{C}}$-orbits $\overline{\mathcal{O}_i}$ in $\mathfrak{p}$, we can write

$$p_M(t) = \sum_{\dim \mathcal{O}_i = \frac{1}{2}GK\dim(\pi)} m(\pi, \mathcal{O}_i)\, p_{R([\overline{\mathcal{O}_i}])}(t) + \text{ lower order terms}$$

  and deduce

$$BernsteinDeg(\pi) = \sum_{\dim \mathcal{O}_i = \frac{1}{2}GK\dim(\pi)} m(\pi, \mathcal{O}_i)\deg\left(\overline{\mathcal{O}}_i\right) \tag{3.2}$$

  where

$$\deg\left(\overline{\mathcal{O}_i}\right) \equiv BernsteinDeg\left(R\left[\overline{\mathcal{O}_i}\right]\right)$$

Remark 3.3. When the characteristic variety of π is the closure of a single $K_{\mathbb{C}}$-orbit $\mathcal{O}$, then we write

$$m(\pi, \mathcal{O}_i) = \frac{BernsteinDeg(\pi)}{BernsteinDeg\left(R\left[\overline{\mathcal{O}}_i\right]\right)} = \lim_{t\to\infty} \frac{\sum_{|\lambda|\leq t} \dim m(\lambda, \pi) \dim V_\lambda}{\sum_{|\lambda|\leq t} \dim m\left(\lambda, R\left[\overline{\mathcal{O}}\right]\right) \dim V_\lambda}$$

(Here $m(\lambda, X)$ is the multiplicity of a K-type with highest weight λ in X, and $\dim V_\lambda$ is the dimension of a K-type with highest weight λ.) This is Trapa's formula.

Remark 3.4. For principal series representations it happens that orbits $\mathcal{O}_i$ appearing in the $K_{\mathbb{C}}$-orbits of maximal dimension in $\mathfrak{p}$. It happens in this case that the degrees of the orbits and their multiplicities all coincide, and so one has

$$m(\pi, \mathcal{O}_i) = \frac{BernsteinDegree(\pi)}{(\#K_{\mathbb{C}}\text{-orbits of maximal dimension})\, Deg(\mathcal{O}_i)}$$

Remark 3.5. The formula (3.2) does not given any information about the multiplicities of the smaller components of the characteristic cycles (corresponding to minimal prime ideals which happen not to correspond to irreducible varieties of maximal dimension in the associated variety of π).

4. The Case at Hand

Let's now return to our problem of computing the characteristic cycles for Sahi's representations π_i. We start with the result, and then fill in the details. But first a little notation.

Recall in our description of the restricted root systems we used a certain basis $\left\{\gamma_j \mid j = 1, \dots n\right\} \in \mathfrak{t}_1^*$. The $\left\{\gamma_j\right\}$ correspond a certain set of strongly orthogonal nilpotent elements of $\mathfrak{p}$; and, in fact, one can choose elements

$$e_j \in \mathfrak{p}_{\gamma_j} \qquad , \qquad f_i \in \mathfrak{p}_{-\gamma_j} \qquad , \qquad h_j \in \left[\mathfrak{p}_{\gamma_j}, \mathfrak{p}_{\gamma_j}\right] \qquad ; \qquad j = 1, \dots, n$$

so that

$$\{\mathfrak{s}_j = \{e_j, f_j, h_j\} \mid j = 1, \dots\}$$

form a set of n mutually commuting standard $\mathfrak{sl}_2$-triples. Set

$$y_i = f_1 + \cdots + f_i$$

and let

$$\mathcal{O}_i \equiv K_{\mathbb{C}} \cdot y_i$$

Theorem 4.1. *The characteristic cycle of a representation π_i is*

$$CC(\pi_i) = \left[\overline{\mathcal{O}_i}\right]$$

In other words, the asscociated variety of π_i consists of the closure of a single $K_{\mathbb{C}}$-orbit, $\overline{\mathcal{O}_i,}$ and that the multiplicity of this orbit in the characteristic cycle is one.

Sketch of proof.

We show first that the associated variety of π_i is precisely $\overline{\mathcal{O}_i}$. With this result we can then apply Trapa's formula,

$$m(\pi, \mathcal{O}_i) = \lim_{t\to\infty} \frac{\sum_{|\lambda|\le t} \dim m(\lambda, \pi) \dim V_\lambda}{\sum_{|\lambda|\le t} \dim m\left(\lambda, R\left[\overline{\mathcal{O}_i}\right]\right) \dim V_\lambda}$$

Let $\mathcal{S}$ be the set of K-types of π_i. Sahi provides a parameterization of these K-types and a proof that they are all multiplicity free, and so

$$m(\pi, \mathcal{O}_i) = \lim_{t\to\infty} \frac{\sum_{\substack{\lambda\in\mathcal{S}\\ |\lambda|\le t}} \dim V_\lambda}{\sum_{|\lambda|\le t} \dim m\left(\lambda, R\left[\overline{\mathcal{O}_i}\right]\right) \dim V_\lambda}$$

Now because $R\left[\overline{\mathcal{O}_i}\right]$ is necessarily a subquotient of $gr(X_{\pi_i})$, each K-type of $R\left[\overline{\mathcal{O}_i}\right]$ must also appear in π_i; hence,

$$m\left(\lambda, R\left[\overline{\mathcal{O}_i}\right]\right) = \begin{cases} 0, 1 & \text{if } \lambda \in \mathcal{S} \\ 0 & \text{if } \lambda \notin \mathcal{S} \end{cases}$$

We will show that in fact every K-type of π_i also appears in $R\left[\overline{\mathcal{O}_i}\right]$, and so

$$m(\pi, \mathcal{O}_i) = \lim_{t\to\infty} \frac{\sum_{\substack{\lambda\in\mathcal{S}\\ |\lambda|\le t}} \dim V_\lambda}{\sum_{\substack{\lambda\in\mathcal{S}\\ |\lambda|\le t}} \dim V_\lambda} = 1$$

5. The Associated Variety of π_i

We first note that the root spaces $\mathfrak{p}_{\gamma_i} \subset \mathfrak{g}$ are commutative since $\gamma_i + \gamma_j$ is not a root of $\mathfrak{g}$. Moreover, for each $j = 1, \dots, n$ we can choose elements $e_j \in \mathfrak{p}_{\gamma j}$, $f_j \in \mathfrak{p}_{\gamma_j}$, and $h_j \in \mathfrak{t}_1$ such that

$$[e_j, f_j] = h_j \quad , \quad [h_j, e_j] = 2e_j \quad , \quad [h_j, f_j] = -2f_j \qquad , \qquad j = 1, \dots, n$$

The aim of this section is to show that the associated variety of (π, V) is the closure of a single $K_{\mathbb{C}}$-orbit. In fact, we shall show that the associated variety of (π, V) is the closure of

$$K_{\mathbb{C}} \cdot \left(\sum_{j=1}^{i} f_j \right)$$

However, we do this indirectly. We will actually show that the real nilpotent orbit corresponding to the right hand side of (2) via the Sekiguchi correspondence is the unique real orbit $\mathcal{O}$ such that $G_{\mathbb{C}} \cdot \mathcal{O}$ is dense in the associated variety of $Ann_{U(\mathfrak{g})}(\pi)$.

The Sekiguchi correspondence is implemented via a Cayley transform defined as follows. Set

$$c_j = \exp\left(\frac{\pi i}{4} (e_j + f_j) \right)$$

and then set

$$E_j = c_j^{-1} e_j c_j \quad , \quad F_j = c_j^{-1} f_j c_j \quad , \quad H_j = c_j^{-1} h_j c_j$$

Lemma 5.1. *Under the Cayley transform, each of the $\mathfrak{sl}_2$ triples $\{e_j, f_j, h_j\} \in \mathfrak{g}_{\mathbb{C}}$ get mapped to an $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}_0$. Moreover, the parabolic subalgebra of $\mathfrak{g}$ corresponding to the span of the non-negative root spaces of the semisimple element*

$$H = H_1 + \cdots + H_n \in \mathfrak{s}_o$$

can be identified with the parabolic subalgebra of G associated with its Jordan algebra structure

Lemma 5.2. $G_{\mathbb{C}} \cdot Y_i = \overline{L_{\mathbb{C}} \cdot Y_i}$.

Step 1. $G_{\mathbb{C}} \cdot Y_i$ is a union of $L_{\mathbb{C}}$-orbits

Step 2. All $L_{\mathbb{C}}$-orbits in $\overline{\mathfrak{n}}$ are of the form $L_{\mathbb{C}} \cdot Y_k$, where $Y_k = \sum_{j=1}^{k} F_j$ and $\overline{L_{\mathbb{C}} \cdot Y_k} \subset L_{\mathbb{C}} \cdot Y_\ell$ if $k \leq \ell$. ([BSZ]).

Step 3. $\overline{L_{\mathbb{C}} \cdot Y_i} \subset G_{\mathbb{C}} \cdot Y_i \cap \overline{\mathfrak{n}}_{\mathbb{C}}$

Since Y_i is stablized by $\overline{N}_{\mathbb{C}}$, and $NL_{\mathbb{C}}\overline{N}_{\mathbb{C}}$ is dense in $G_{\mathbb{C}}$, $N_{\mathbb{C}}L_{\mathbb{C}} \cdot Y_i \cap \overline{\mathfrak{n}}_{\mathbb{C}}$ is dense in $G_{\mathbb{C}} \cdot F_i \cap \overline{\mathfrak{n}}_{\mathbb{C}}$. Since $[\overline{\mathfrak{n}}, \mathfrak{n}] \subset \mathfrak{l}$, $[\mathfrak{n}, \mathfrak{l}] \subset \mathfrak{n}$,

$$N_{\mathbb{C}}L_{\mathbb{C}} \cdot X_i \cap \overline{\mathfrak{n}}_{\mathbb{C}} = L_{\mathbb{C}} \cdot Y_i$$

Hence, $L_{\mathbb{C}} \cdot Y_i$ is dense in $G_{\mathbb{C}} \cdot Y_i \cap \overline{\mathfrak{n}}_{\mathbb{C}}$.

Step 4. We now show that the action of $G_{\mathbb{C}}$ can not bump Y_i to a larger orbit $L_{\mathbb{C}}$-orbit. Assume $\overline{L_{\mathbb{C}} \cdot Y_\ell} = G_{\mathbb{C}} \cdot Y_i \cap \overline{\mathfrak{n}}_{\mathbb{C}}$, and $\ell > i$.

$$\begin{aligned} &\Rightarrow \qquad L_{\mathbb{C}} \cdot Y_\ell \cap N_{\mathbb{C}}L_{\mathbb{C}} \cdot Y_i \cap \overline{\mathfrak{n}}_{\mathbb{C}} \neq \varnothing \\ &\Rightarrow \qquad \exists\, n \in N_{\mathbb{C}} \ ,\ l \in L_{\mathbb{C}} \text{ such that } Y_\ell = Ad(n)\, Ad(l)\, Y_i \end{aligned}$$

$\mathfrak{n}_{\mathbb{C}}$ is abelian, so $n = \exp(x)$ and

$$Y_\ell = Ad(l)(Y_i) + [x, Ad(l)\, Y_i] + \frac{1}{2}[x, [x, Ad(l)\, Y_i]$$

Now $Y_\ell \in \overline{\mathfrak{n}}_{\mathbb{C}}$ and $Ad(l)(Y_i) \in \overline{\mathfrak{n}}_{\mathbb{C}}$, but $[x, Ad(l)\, Y_i] \in \mathfrak{l}_{\mathbb{C}}$,and $\frac{1}{2}[x, [x, Ad(l)\, Y_i] \in \mathfrak{n}_{\mathbb{C}}$. So

$$\overline{L_{\mathbb{C}} \cdot Y_\ell} = G_{\mathbb{C}} \cdot Y_i \cap \overline{\mathfrak{n}}_{\mathbb{C}}, \quad \text{and} \quad \ell > i. \quad \Rightarrow \quad Y_\ell = Ad(l)\, Y_i \text{ with } l \in L_{\mathbb{C}}$$

Lemma 5.3. *If $\mathcal{O}$ is a $G_{\mathbb{R}}$-orbit of top dimension in $G_{\mathbb{C}} \cdot Y_i$, then $\mathcal{O} \cap \mathfrak{n} \neq \varnothing$.*

Proof. Since $\mathcal{O}$ has top dimension,

$$\begin{aligned}
\mathcal{O}\cap N_{\mathbb{C}}L_{\mathbb{C}}\cdot Y_i &\neq \varnothing \\
&\Rightarrow \quad \exists\, z\in\mathcal{O}\cap N_{\mathbb{C}}L_{\mathbb{C}}\cdot Y_i \\
&\Rightarrow \quad \exists\, n_x=\exp x\in N_{\mathbb{C}}\ ,\ l\in L_{\mathbb{C}} \text{ such that } z=Ad\,(n_x)\,Ad\,(l)\,Y_i \text{ and } z\in\mathfrak{g}_{\mathbb{R}}
\end{aligned}$$

Then

$$z=Ad\,(l)\,Y_i+[x,Ad\,(l)\,Y_i]+\frac{1}{2}\,[x,[x,Ad\,(l)\,Y_i]]$$

Now $Ad\,(l)\,(Y_i)\in\overline{\mathfrak{n}}_{\mathbb{C}}$, but $[x,Ad\,(l)\,Y_i]\in\mathfrak{l}_{\mathbb{C}}$,and $\frac{1}{2}\,[x,[x,Ad\,(l)\,Y_i]\in\mathfrak{n}_{\mathbb{C}}$ and all three terms on right hand side have to reside in $\mathfrak{g}_{\mathbb{R}}$, $l\in L\subset L_{\mathbb{C}}$,and $x\in\mathfrak{n}$.Thus,

$$z=Ad\,(n)\,Ad\,(l)\,Y_i$$

is such that $Ad\left(n^{-1}\right)z=Ad\,(l)\,Y_i\in\overline{\mathfrak{n}}$, and also lies in $\mathcal{O}$.

Lemma 5.4. *If $\mathcal{O}$ is a real G-orbit contained in $G_{\mathbb{C}}\cdot Y_i$, with $\dim_{\mathbb{R}}(\mathcal{O})=\frac{1}{2}\dim_{\mathbb{C}}G_{\mathbb{C}}\cdot Y_i$, then $\mathcal{O}=G\cdot Y_i$.*

Proof. Assume $\mathcal{O}$ is a real G orbit contained in $G_{\mathbb{C}}\cdot Y_i$ and that $\dim_{\mathbb{R}}(\mathcal{O})=\dim_{\mathbb{C}}G_{\mathbb{C}}\cdot Y_i$. By the preceding lemma $\mathcal{O}\cap\overline{\mathfrak{n}}$ is non-empty. It is also L-invariant, and so a union of L-orbits. In [BSZ], it is shown[2] that each of the L-orbits in $\overline{\mathfrak{n}}$ is of the form, $L\cdot Y_k$. Moreover, if $k<\ell$, the stabilizer of Y_ℓ in $\mathfrak{l}$ is strictly contained within the stabilizer of Y_k in $\mathfrak{l}$, and so $\dim(L\cdot Y_k)<\dim(L\cdot Y_\ell)$ if $k<\ell$. Thus, there will be at most one L-orbit of any particular dimension in $\mathcal{O}\cap\overline{\mathfrak{n}}$. Since $\mathcal{O}_i=G\cdot Y_i\subset G_{\mathbb{C}}\cdot Y_i$ and $\dim_{\mathbb{R}}(\mathcal{O}_i)=\frac{1}{2}\dim_{\mathbb{C}}G_{\mathbb{C}}\cdot Y_i$, the lemma follows. .

Lemma 5.5. $GK\dim\pi_i=\frac{1}{2}\dim_{\mathbb{C}}G_{\mathbb{C}}\cdot Y_i$

Proof. This is proved by computing the dimension of the stabilizer of Y_i in $\mathfrak{g}_{\mathbb{C}}$ and then comparing it with the Gelfand-Kirilov dimension of π_i (as computed in previous section.).

Lemma 5.6. $Ass\left(Ann_{U(\mathfrak{g})}(\pi_i)\right)=\overline{G_{\mathbb{C}}\cdot Y_i}$

Proof. This can be proved by using the explicit realization of [BSZ] to show that $Y_i\in Ass\left(Ann_{U(\mathfrak{g})}(\pi_i)\right)$.

Alternatively, once one can show that $\exp^{B(Y_i,\cdot)}$ defines a character for a Whittaker vector for π_i, and then by a theorem of Matumoto $G_{\mathbb{C}}\cdot Y_i\subset Ass\left(Ann_{U(\mathfrak{g})}(\pi_i)\right)$. Comparing dimensions the conclusion follows.

Lemma 5.7. *Choose $f_j\in\mathfrak{s}_{\gamma_j}$ as in the beginning of this section, and set*

$$y_i=f_1+\cdots f_i\in\mathfrak{s}_{\mathbb{C}}$$

Then the associated variety $\mathcal{V}(\pi_i)\subset\mathfrak{s}$ of π_i is the Zariski closure of $K_{\mathbb{C}}\cdot y_i$.

Proof. Set y_i is just the inverse Cayley transform of Y_i, and so this lemma just implements the Kostant-Sekiguchi correspondence between G orbits.

Henceforth, we'll let denote by $\mathcal{O}_i$ the unique dense orbit $K_{\mathbb{C}}\cdot y_i$ in the associated variety of π_i.

Corollary 5.8. *The characteristic cycle of π_i is of the form*

$$\mathcal{V}(\pi_i)=m\left(\pi_i,\overline{\mathcal{O}}_i\right)\left[\overline{\mathcal{O}}_i\right]$$

Proof. We have seen that associated variety of $Ann_{U(\mathfrak{g})}(\pi_i)$ is $\overline{G_{\mathbb{C}}\cdot Y_i}$, and that inside $\overline{G_{\mathbb{C}}\cdot Y_i}$ there is only one real nilpotent orbit whose dimension is equal to $GK\dim(\pi_i)$, and that orbit is simply $G\cdot Y_i$. The Kostant-Sekiguchi corrrespondence then implies that there is only one $K_{\mathbb{C}}$-orbit in $\mathfrak{p}$ that lies in the

[2]Actually, in [BSZ] it is shown that the L-orbits in $\mathfrak{n}$ are of the form $L\cdot(E_1+\cdots+E_k)$, but their argument is readily ported to yield the analogous classification of the L-orbits in $\overline{\mathfrak{n}}$.

associated variety of $Ann_{U(\mathfrak{g})}(\pi_i)$ of complex dimension $\frac{1}{2}GK\dim(\pi_i)$, and that orbit is $K_{\mathbb{C}}\cdot y_i$. Theorem 8.4 of Vogan[3] now implies that the characteristic variety of π_i must be $\overline{\mathcal{O}}_i \equiv \overline{K_{\mathbb{C}}\cdot y_i}$.

6. $m\left(\pi_i,\overline{\mathcal{O}_i}\right)=1$

In our sketch of the proof of Theorem 4.1, the proposition that $m\left(\pi_i,\overline{\mathcal{O}}_i\right)=1$ followed from (the applicability of) Trapa's formula and the fact that every K-type in π_i also appears in $R\left[\overline{\mathcal{O}}_i\right]$. We'll now prove the latter.

Recall that we have a set of n strongly orthogonal non-compact roots and a corresponding set of n commuting $\mathfrak{sl}_2$-triples $\{e_i,f_i,h_i\}$ with $e_i\in\mathfrak{p}_{\gamma_i}$, $f_i\in\mathfrak{p}_{-\gamma_i}$.

Lemma 6.1 (Sahi)**.** *Let ϕ_o be the unique spherical vector in π_i. Then, for $j=1,2,\dots,i$*

$$e_1e_2\cdots e_j\phi_o$$

has a non-zero projection onto the K-type $V_{\gamma_1+\gamma_2+\cdots+\gamma_j}$ of π_i.

Corollary 6.2. *For each $i=1,\dots,n$ there is an irreducible summand V_{λ_i} of highest weight $\lambda_i=\gamma_1+\gamma_2+\cdots+\gamma_i$ in $S^i(\mathfrak{p})$.*

Proof. The e_j all commute with one another, and so the products $e_1e_2\cdots e_i$ can be identified with the corresponding monomials in $S(\mathfrak{p})$. Indeed, they can regarded as the images in $U(\mathfrak{p})$ of the monomials $e_1\cdots e_i\in S^i(\mathfrak{p})$ via the symmeterizer map $sym: S(\mathfrak{g})\to U(\mathfrak{g})$. Following the symmeterizer map with Sahi's projection, we have a K-equivariant map that sends $e_1\cdots e_i$ to the K-type $V_{\gamma_1+\cdots+\gamma_i}$ of π_i. It follows that there must be a summand $V_{\lambda_i}\in S^i(\mathfrak{p})$ of highest weight λ_i.

Corollary 6.3. *The monomial $\phi_i=e_1\cdots e_i\in S^i(\mathfrak{p})$ has a non-zero projection onto V_{λ_i}.*

Lemma 6.4. *The summand $V_{\lambda_j}\subset S^j(\mathfrak{p})$ does not vanish on $\mathcal{O}_i$ if $j\le i$.*

Proof. The monomial ϕ_i has non-zero projection onto V_{λ_i} and in fact it must project onto the highest weight vector of V_{λ_i}. However, ϕ_i itself need not (and is not unless $i=1$) a highest weight vector. Let $e_{n+1},\dots,e_{\dim\mathfrak{p}}$ a basis for the complement of $span(e_1,\dots,e_n)$ in $\mathfrak{p}$, chosen in such a way that each e_i is a weight vector for K. Then the highest weight vector ψ_j for $V_{\lambda j}$ will be of the form

$$\begin{aligned}\psi_j &= e_1\cdots e_j+\text{ sum of monomial terms of degree } i \text{ and weight } \lambda_i\\ &= \phi_j+\sum\varphi_{j,k}\end{aligned}$$

Now recall that $\mathcal{O}_i=K_{\mathbb{C}}\cdot y_i$. Evaluating ψ_j are the base point $y_i=f_1+\cdots+f_i$ (and employing the Killing form to evaluate polynomials on $\mathfrak{p}$ on $y_i\in\mathfrak{p}$), one obtains

$$\psi_j(y_i)=\phi_j(y_i)+\sum\varphi_{j,k}(y_i)$$

Now

$$\begin{aligned}\phi_j(y_i) &= \langle e_1,f_1+\cdots+f_i\rangle\cdots\langle e_j,f_1+\cdots+f_i\rangle\\ &= \langle e_1,f_i\rangle\cdots\langle e_j,f_j\rangle\\ &\neq 0\end{aligned}$$

[3]Theorem: Let (G,K) be a reductive symmetric pair of Harish-Chandra class, and X an irreducible $(\mathfrak{g},K)$-module. Write $J=Ann_{U(\mathfrak{g})}X$, a primitive ideal in $U(\mathfrak{g})$. Let $\mathcal{O}\subset\mathcal{V}(J)$ be the dense G-orbit in $\mathfrak{g}^*$, and let $\mathcal{V}(X)\subset(\mathfrak{g}/\mathfrak{k})^*$ be the associated variety of X. Then

(a) $\mathcal{V}(X)\subset\mathcal{V}(J)\cap(\mathfrak{g}/\mathfrak{k})^*$
(b) $\mathcal{O}\cap(\mathfrak{g}/\mathfrak{k})^*$ is a finite union of $K_{\mathbb{C}}$-orbits $\mathcal{O}_1,\dots,\mathcal{O}_r$, each of which has (complex) dimension equal to half that of $\mathcal{O}$.
(c) Some of the $\mathcal{O}_i$ are contained in $\mathcal{V}(X)$, they are precisely the K-orbits of maximal dimension in $\mathcal{V}(X)$.

On the other hand, each of the monomial terms $\varphi_{j,k}$ will contain factors corresponding to coordinates that evaluate to zero on y_i. And so

$$\psi_j(y_i) = \phi_j(y_i) \neq 0.$$

Having shown that the highest weight vector of V_{λ_j} does not vanish on $\mathcal{O}_i$ the lemma follows.

Corollary 6.5. *For every K-type $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$ occuring in π_i there is a corresponding summand $V_{\alpha_1\gamma_1+\cdots+a_i\gamma_i} \subset S(\mathfrak{p})$ supported on $\mathcal{O}_i$. Hence, each K-type of π_i occurs in $R\left[\overline{\mathcal{O}}_i\right]$.*

Proof. By the preceding lemma, the highest weight vectors ψ_j of $V_{\gamma_1+\cdots+\gamma_j} \subset S^j(\mathfrak{p})$ are supported at $y_i \in \mathcal{O}_i$. But then by forming products of the form

$$(\psi_i)^{m_1} \cdots (\psi_i)^{m_i}$$

such that

$$\begin{aligned} a_1 &= m_1 + \cdots m_i \\ a_2 &= m_2 + \cdots + m_i \\ &\vdots \\ a_i &= m_i \end{aligned}$$

we can create a highest weight vector of a summand $V_{a_1\gamma_1+\cdots+a_i\gamma_i} \subset S(\mathfrak{p})$ that does not vanish at y_i. Thus the K-type $\lambda = a_1\gamma_1 + \cdots + a_i\gamma_i$ appears in $R\left[\overline{\mathcal{O}}_i\right]$.