A taxonomy of irreducible Harish-Chandra modules of regular integral infinitesimal character

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1. Introduction

Let G be a reductive Lie group, K a maximal compact subgroup of G. In fact, we shall assume that G is a set of real points of a linear algebraic group $\mathbb{G}$ defined over $\mathbb{C}$. Let $\mathfrak{g}$ be the complexified Lie algebra of G, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a corresponding Cartan decomposition of $\mathfrak{g}$

Definition 1.1. *A $(\mathfrak{g}, K)$-**module** is a complex vector space V carrying both a Lie algebra representation of $\mathfrak{g}$ and a group representation of K such that*

- *The representation of K on V is locally finite and smooth.*
- *The differential of the group representation of K coincides with the restriction of the Lie algebra representation to $\mathfrak{k}$.*
- *The group representation and the Lie algebra representations are compatible in the sense that*

$$\pi_K(k)\,\pi_{\mathfrak{k}}(X) = \pi_{\mathfrak{k}}(Ad(k)\,X)\,\pi_K(k) \quad .$$

Definition 1.2. *A (Hilbert space) representation π of a reductive group G with maximal compact subgroup K is called **admissible** if $\pi|_K$ is a direct sum of finite-dimensional representations of K such that each K-type (i.e each distinct equivalence class of irreducible representations of K) occurs with finite multiplicity.*

Definition 1.3 (Theorem)**.** *Suppose (π, V) is a smooth representation of a reductive Lie group G, and K is a compact subgroup of G. Then the space of K-finite vectors can be endowed with the structure of a $(\mathfrak{g}, K)$-module. We call this $(\mathfrak{g}, K)$-module the **Harish-Chandra module** of (π, V).*

Theorem 1.4 (Langlands Classification)**.** *Fix a minimal parabolic subgroup $P_o = M_oA_oN_o$. Then equivalence classes of irreducible admissible representations of G are in a one-to-one correspondence with the set of triples $(P, [\sigma], \nu)$ such that*

- *P is a parabolic subgroup of G containing P_o*
- *σ is an equivalence class of ireducible tempered unitary representations of M and $[\sigma]$ is its equivalence class*
- *ν is an element of $\mathfrak{a}'$ with $Re(\nu)$ in the open positive chamber.*

The correspondence is

$$(P, [\sigma], \nu) \longleftrightarrow \textit{unique irreducible quotient of } Ind_{MAN}^{G}(\sigma \otimes e^{\nu} \otimes 1)$$

Over that past thirty years there has been a succession of reparameterizations of Langland's parameters; via results of Knapp and Zuckerman "tempered representations" can be replaced by "discrete series representations or limits of discrete series representations"; then, to make connections with Shelstad's notion of "endoscopic transfer" and the Arthur conjectures, Adams and Vogan "upgraded" the Langlands' parameterization to a parameterization in terms of L-groups. And now, in the Atlas program, there is combinatorial version of "Langland's parameters" in terms of pairs (x, y) of "L-data". In this latter picture the L-datum x corresponds to a triple (η, B, λ) where η is a "strong involution" for $\mathbb{G} = (G)_{\mathbb{C}}$, B is a Borel subgroup of $\mathbb{G}$, λ is an element of the weight lattice of $\mathbb{G}$ with respect to a Cartan subgroup $H \subset B \subset \mathbb{G}$, and y is an L-datum for the dual group $\mathbb{G}^{\vee}$ of $\mathbb{G}$. How one actually connects these L-data pairs with the "Langlands parameters" of the theorem above is a very tricky business; but, roughly speaking, the first element x of the pair fixes the L-packet containing the representation, the second element fixes the R-packet containing the representation, and the representation (x, y) is the unique representation lying in the intersection of this L-packet and this R-packets. The purpose of this talk is to describe some recent progress in breaking up the set $\widehat{G}_{adm}$ of irreducible admissible representations into subsets sharing a strong and stronger algebraic invariant. I should point out that the principal contributors in this business, to my knowledge at least, have been David Vogan, Monty McGovern, Peter Trapa, and myself; but there might be others as well since it not always clear as who has talked to who.

2. Reduction to Infinitesimal Character ρ

The first and easiest partitioning of set of irreducible admissible representations is by grouping together those representations with the same infinitesimal character; that is, we group together those irreducible admissible representations for which the action of the center of $U(\mathfrak{g})$ on the corresponding Harish-Chandra module is by the same character. We shall denote by HC_λ the full subcategory of Harish-Chandra modules of infinitesimal character λ.

Now, it turns out that the invariants we shall examine are also well-behaved with respect to tensoring by finite-dimensional representations and so, for example, results for the irreducible Harish-Chandra modules of infinitesimal character ρ, can be "translated" to the entire coherent family containing HC_ρ; i.e, that is, via the Zuckerman translation principle results for HC_ρ can extended to the set of irreducible Harish-Chandra modules of regular integral infinitesimal character. So henceforth we shall presume to be working in HC_ρ. And so doing, we can now think of the pairs (x, y) specifying an irreducible Harish-Chandra module as corresponding to pairs of strong involutions (one for G and one for a dual real form of G), or a bit more geometrically, a element of $K\backslash G/B$ and an element of $K^\vee\backslash G^\vee/B^\vee$.

3. Blocks, cells and KLV polynomials

3.1. **Blocks of irreducible Harish-Chandra modules.** Because of the obvious duality in the parameterization it is natural, in the atlas point-of-view, to let x and y separately vary over all the strong involutions of all the strong real forms of (fixed inner classes of) respectively, $\mathbb{G}$ and $\mathbb{G}^\vee$. (Of course, when one is concerned with **only** the irreducible admissible representations of a particular real form, one need only vary y over the strong involutions of the dual group.)

At any rate, viewing the parameters x and y as being "attached" to, respectively, a real form of $G_{\mathbb{C}}$ and real form of $G_{\mathbb{C}}^\vee$, we immediately arrive at an initial, albeit atlas-centric, partitioning of the set of irreducible admissible representations G; we simply group together those representations (x, y) where x is a strong involution corresponding to our fixed real form G of $G_{\mathbb{C}}$ and y corresponds to a fixed real form of $\mathbb{G}^\vee$. Thus, a *block* of representations of G is simply a set of representations (x, y) where the strong involutions y correspond to a particular (equivalence class of real) form(s) of $(G_{\mathbb{C}})^\vee$. Below is a table is listing the number of elements in each "block"' of E_8:

	e_8	$E_8(e_7, su(2))$	$E_8(\mathbb{R})$
e_8 (compact)	0	0	1
$E_8(e_7, su(2))$ (quaternionic)	0	3150	73410
$E_8(\mathbb{R})$ (split)	1	73410	453060

The total number of equivalences classses irreducible Harish-Chandra modules of the split form $E_8(\mathbb{R})$ with infinitesimal character ρ is thus

$$1 + 73410 + 453060 = 526\,471$$

and these fall into three blocks, corresponding to the three real forms of $E_8(\mathbb{C})^\vee \approx E_8(\mathbb{C})$.

3.2. **KLV-polynomials.** Consider the Grothendieck group $\mathcal{G}_{HC}$ of the category of Harish-Chandra modules; that is, the set of formal $\mathbb{Z}$-linear combinations irreducible Harish-Chandra modules J_λ. Well, actually, the set of *irreducible* Harish-Chandra modules is only a particular basis for $\mathcal{G}_{HC}$. The set of *standard* Harish-Chandra is an equally viable basis. The KLV polynomials provide a means of translating from one basis to the other.

Recall that the *standard* Harish-Chandra module M_λ corresponding to a Langland's parameter $\lambda = \sigma \otimes e^\nu \otimes 1$ is just the Harish-Chandra module of the full induced representation

$$M_\lambda = Ind_P^G (\sigma \otimes e^\nu \otimes 1)$$

while the *irreducible* Harish-Chandra module J_λ is unique maximal quotient of M_λ. Passing to the Grothendieck group, we can write

$$J_\lambda = M_\lambda + m_{\lambda\nu_1} M_{\nu_1} + \cdots + m_{\lambda\nu_k} M_{\nu_k}$$

We note that the standard representation M_λ containing J_λ always appears with coefficient 1 and the rest of the expansion is of finite-length and is in a certain natural sense "upper-triangular".

In fact, the coefficients $m_{\lambda\nu_i}$ are determined by the Kahzdan-Lusztig conjecture (now a theorem).

Theorem 3.1 (KL conjecture for real groups)**.** *There exists a set of polynomials $P_{\lambda\nu}(q)$, indexed by Langlands parameters (with the same infinitesimal character) such that in the expansion*

$$J_\lambda = m_{\lambda\lambda} M_\lambda + m_{\lambda,\nu_1} M_{\nu_1} + \cdots + m_{\lambda\nu_k} M_{\nu_k}$$

the coefficients m_{λ,ν_i} are such that

$$m_{\lambda,\nu_i} = (-1)^{\ell(\lambda)-\ell(\nu_i)} P_{\nu_i,\lambda}, (1)$$

The polynomials $P_{\lambda\nu}(q)$ are the so-called Kahzdan-Lusztig-Vogan polynomials. The big announcement last March was the explicit computation of the KLV-polynomials for E_8.

3.3. **Cells of Harish-Chandra modules.** Now the utility of the KLV-polynomials is by no means limited to simply interpolating between the standard representations and irreducible representations. In fact, upon inversion, the coefficients of the various powers of q prescribe the multiplicities in a Janzten filtration of M_λ. The following theorem is the basis of our next refinement of HC_λ.

Theorem 3.2. *Suppose J_x, J_y are irreducible H-C modules of infinitesimal character ρ. Then the multiplicity of J_y in $J_x \otimes \mathfrak{g}|_\rho$ is exactly the coefficient of $q^{(|x|-|y|-1)/2}$ in $P_{x,y}(q)$.*

To put this to use we now introduce the notion of *cells* of Harish-Chandra modules.

Definition 3.3. *Given two objects X, Y in HC_λ, we say $X \rightsquigarrow Y$ if there exists a finite-dimensional representation F of G appearing in the tensor algebra $T(\mathfrak{g})$ such that Y appears as a subquotient of $X \otimes F$. Write $X \sim Y$ if $X \rightsquigarrow Y$ and $Y \rightsquigarrow X$. The equivalence classes for the relation $\sim$ are called **cells** (of Harish-Chandra modules).*

It turns out that the decomposition of HC_λ into disjoint cells is compatible with the decomposition into blocks, and moreover, given the preceding theorem, this decomposition is explicitly computable via the preceding theorem and knowledge of the KLV-polynomials. In fact, the Atlas software produces this decomposition as a *by-product* of its computation of the KLV-polynomials.

3.4. **Wgraphs.**

Definition 3.4. *Consider a block B of irreducible Harish-Chandra modules of infinitesimal character ρ. The W-**graph** of B is the weighted graph where:*

- *the vertices are the elements $v \in B$*

- *there is an edge* (v, v') *of* ***multiplicity*** m *between two vertices if*

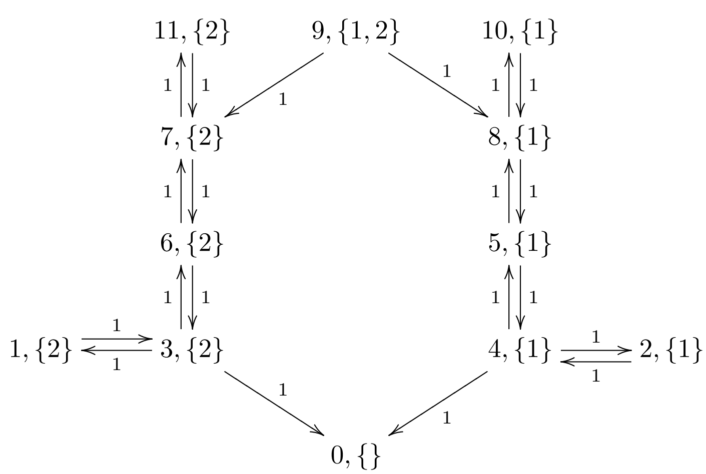
$$\textit{coefficient of } q^{\left(|v|-|v'|-1\right)/2} \textit{ in } P_{v,v'}\left(q\right) = m \neq 0$$

- *there is assigned to each vertex* v *a subset* $\tau\left(v\right)$ *of the set of simple roots of* $\mathfrak{g}$ *(the* ***descent set*** of *v).*

Remark 3.5. All of this data is contained in the output of the Atlas `wgraph` command. Below is an example of the (annotated) output of `wgraph` for the big block of G_2.

```
block element descent set (edge vertex,multiplicity)
      0           {}              {}
      1           {2}           {(3,1)}
      2           {1}           {(4,1)}
      3           {1}     {(0,1),(1,1),(6,1)}
      4           {2}     {(0,1),(2,1),(5,1)}
      5           {1}        {(4,1),(8,1)}
      6           {2}        {(3,1),(7,1)}
      7           {1}        {(6,1),(11,1)}
      8           {2}        {(5,1),(10,1)}
      9          {1,2}       {(7,1),(8,1)}
      10          {1}           {(8,1)}
      11          {2}           {(7,1)}
```

The W-graph for this block thus looks like



Note that there are four cells: $\{0\}$, $\{1, 6, 7, 11\}$, $\{2, 4, 5, 8, 10\}$, and $\{9\}$. We remark it is not necessary that cell elements be connected by linked by double edges, two vertices v, v' will belong to the same cell if and only if there is a directed path from v to v' and a directed path from v' to v.

Remark 3.6. The representations in the same cell have the same Gelfand-Kirillov dimension (as they are connected via a tensoring by a finite dimensional representation). In fact, they have the same associated variety. So how come we are finding only three cells when there are five nilpotent orbits for G_2? The answer will come in the next installment when we connect cells with special nilpotent orbits. To prepare for that discussion, I'll conclude today's seminar with a brief digression into Weyl group cells and primitive ideal theory.

4. Primitive Ideals and Cells in W

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Let V be a left $U(\mathfrak{g})$-module. The **annihilator** of V is the two-sided ideal $Ann(V)$ defined by

$$Ann(V) = \{x \in U(\mathfrak{g}) \mid xv = 0 \ , \ \forall\ v \in V\}$$

A **primitive ideal** is the annihilator of an irreducible $U(\mathfrak{g})$-module. Let $Prim(\mathfrak{g})$ denote the set of primitive ideals of $U(\mathfrak{g})$.

On the other hand, if V is an irreducible $U(\mathfrak{g})$-module, the center of $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by a character χ_λ; and in fact,

$$Ann(V) \cap Z(\mathfrak{g}) = ker\chi_\lambda$$

(Here we are thinking of the characters χ_λ as being parameterized by W-orbits λ in $\mathfrak{h}^*$ via the Harish-Chandra isomorphism.). Collecting together the set of primitive ideals having the same infinitesimal character we have

$$Prim(\mathfrak{g}) = \coprod_{\lambda \in \mathfrak{h}^*/W} Prim(\mathfrak{g})_\lambda$$

where

$$Prim(\mathfrak{g})_\lambda = \{\text{primitive ideals with central character } \lambda\}$$

As a hint as to what is ahead, we note that every irreducible Harish-Chandra module will also have an associated infinitesimal character, an associated variety and an associated primitive ideal and our principal achievement over the last summer has been the explicit splitting of the irreducible Harish-Chandra modules of infinitesimal ρ into subsets with the same associated variety and even finer subsets with the same infinitesimal character.

However, it turns out that primitive ideal theory is most easily understood in terms of Verma modules, rather than Harish-Chandra modules.

Definition 4.1. *Let $\lambda \in \mathfrak{h}^*$. The **Verma module** $M(\lambda)$ of highest weight $\lambda + \rho$ is the left $U(\mathfrak{g})$-module*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda+\rho}$$

Here $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$, and $\mathbb{C}_{\lambda+\rho}$ is the 1-dimensional representation of $\mathfrak{b}$ defined by

$$(h + x)\, v = (\lambda + \rho)(h)\, v \quad \forall\ h \in \mathfrak{h} \ , \ x \in \mathfrak{n} \ , \ v \in \mathbb{C}_{\lambda+\rho}$$

Theorem 4.2 (Duflo)**.** *For $w \in W(\mathfrak{g}, \mathfrak{h})$ set*

$$M_w = M(w\rho - \rho)$$

and let

$$L_w = \textit{unique irreducible quotient of } M_w$$

Then

$$\varphi : W \to Prim(\rho) : w \to Ann(L_w)$$

is a surjection.

Definition 4.3. *Let $\sim$ be the equivalence relation on W defined by*

$$w \sim w' \quad \Longleftrightarrow \quad \varphi(w) = \varphi(w')$$

*The corresponding equivalence classes of elements of W are called **left cells** in W.*

To define double cells and special representations we need a technical device due to Joseph.

Theorem 4.4. *For $\mu \in \mathfrak{h}^*$ set*

$$p(\mu) = GoldieRank(U(\mathfrak{g})/Ann(L(\mu)))$$

Then for $w \in W$, $p(w\mu)$ is a harmonic polynomial on $P(\Delta)^{++}$ (the dominant regular chamber). Moreover, if w, w' belong to the same left cell $\mathcal{C}$

$$p_{w'} = p_w \quad (\equiv p(w\rho))$$

Since each Goldie rank polynomial p_w is a harmonic polynomial on $\mathfrak{h}^*$, W can act p_w and thereby generates an irreducible finite-dimensional representation of W.

Definition 4.5. *Fix a finite-dimensional representation σ of W. The $w \in W$ such*

$$\mathbb{C}\langle W \cdot p_w \rangle \approx \sigma$$

comprise the ***double cel****l in W corresponding to $\sigma \in \widehat{W}$. The representations of W that arise in this fashion are called* ***special representations*** *of W.*

Theorem 4.6.

$$\#Prim(\mathfrak{g})_\rho = \sum_{\sigma \in \widehat{W}_{special}} \dim \sigma$$

4.1. **Special Nilpotent Orbits.** Remarkably, the special representations of the Weyl group arise in quite a different way, in connection with the Springer correspondence. The way one attaches a Weyl group representation to a nilpotent orbit is even more complicated than Goldie polynomial method for primitive ideals. Unlike the case of primitive ideals though, this construction does not always attach an irreducible representations of W to an orbit $\mathcal{O}$. However, by tossing out the nonspecial representations, it does always attach a unique special representation to $\mathcal{O}$. But the easiest method of defining special nilpotent orbits is the following: the special nilpotent orbits are precisely those nilpotent orbits that occur as the associated variety of an irreducible $U(\mathfrak{g})$-module with regular integral infinitesimal character.

5. Next Time

Next time I'll describe how two remarkable discoveries of mine allow one to determine when two block elements $i \sim (x, y)$ and $j \sim (x', y')$ share the same associated variety, and when they share the same primitive ideal.