A taxonomy of irreducible Harish-Chandra modules of regular integral infinitesimal character, II

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1. Recap

1.1. **HC$_\rho$.** Last time I introduced the category of study, the Harish-Chandra modules of the irreducible admissible representations of regular integral infinitesimal character and immediately applied the Zuckerman translation principal to reduce the analysis of the associated varieties and primitive ideals of the objects in this set to the analysis of the case of irreducible Harish-Chandra modules of infinitesimal character ρ. We denoted the set of (equivalence classes of) irreducible Harish-Chandra modules of infinitesimal character ρ by HC_ρ.

1.2. **Blocks.** I then described how the Atlas version of the Langlands parameterization immediately breaks up HC_ρ into block. This happens because at infinitesimal character ρ the Atlas parameters (x, y) amount to a choice of a $K\backslash\mathbb{G}/B$ orbit x and a choice of a $K^\vee\backslash\mathbb{G}^\vee/B^\vee$ orbit y. Here $K \subset \mathbb{G}$ is the complexification of the maximal compact subgroup of the real linear group G of interest, and $K^\vee$ is the complexification of the maximal compact subgroup of a real form of the dual group of $\mathbb{G} = G_\mathbb{C}$; and $B, B^\vee$ are fixed Borel subgroups of, respectively, $\mathbb{G}$ and $\mathbb{G}^\vee$. The blocks in HC_ρ corresponded to

$$\{(x, y) \in HC_\rho \mid y \in K^\vee\backslash\mathbb{G}^\vee/B^\vee, \text{ corresponding to a fixed real form of } \mathbb{G}^\vee \ \}$$

1.3. **Cells.** We defined a W-graph $\mathcal{G}_B$ of a block as follows: the vertices of $\mathcal{G}_B$ are simply the elements $i \longleftrightarrow (x, y)$ of the block (which we henceforth consider to be enumerated by integers $i \in \{0, \dots, |B| - 1\}$). We draw an directed edge between two vertices i and j if

$$i \to j \quad : \pi_j \text{ occurs in } \pi_i \otimes \mathfrak{g}$$

The cells of a block B are the groups of vertices C for which there is a directed path (i.e., a sequence of directed edge traversals) connecting i to j, as well directed path connecting j to i.

Since the representations occuring in $\pi_i \otimes \mathfrak{g}$ can not have Gelfand-Kirillove dimension greater than that of π_i, it is clear that the representations in a given cell all have the same Gelfand-Kirillov dimension. In fact,

Theorem 1.1. *The representations in a given cell all have the same associated variety.*

Different cells can have the same associated variety, however. And in fact, until recently we did not even know how to identify which nilpotent orbit has $AV\left(Ann\left(\pi_i\right)\right) = \overline{G_\mathbb{C} \cdot AV\left(\pi_i\right)}$ as its closure. However, this now can be done and I give below the results for the cells in the $E_8\left(\mathbb{R}\right)$ block of $E_8\left(\mathbb{R}\right)$. Sometime later I will describe the mathematics underlying this computation (which involves a serious digression into Spaltenstein duality, and parabolic induction of nilpotent orbits, and the introduction of the idea of the *tau set* of an orbit).

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B-C Type      Cell #'s
E8            0
E8(a1)        1
E8(a2)        2
E8(a3)        3,5
E8(a4)        4,7
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E8(b4)         6,8,9
E8(a5)         10,13,17
E7(a1)         11
E8(b5)         12,14
E8(a6)         15,18
D7(a1)         16,19,20,21,22
E8(b6)         23,25,27
E7(a3)         24,29
E6(a1)+A1      26,30
D7(a2)         28,31,35
E6             33
D5+A2          32,38,39,45
E6(a1)         34,37
E7(a4)         36,41,43
A6+A1          40
D6(a1)         44,49
A6             42,46,48,50
E8(a7)         47,51,53 (the self-dual special orbit and the corresponding self-dual cells)
D5             54
E6(a3)         55,63
D4+A2          52,57,59,62
A4+A2+A1       60
D5(a1)+A1      58,64,65
A4+A2          56,66,68,69
A4+2A1         61,67,70
D5(a1)         71,79
A4+A1          72,80
D4(a1)+A2      73,78,81
A4             74,82
A3+A2          75,76,83,84,86
D4(a1)+A1      87,89
D4             85
D4(a1)         88,90
2A2            77,91,92
A3             94
A2+2A1         93,95,96
A2+A1          97,99
A2             98,100
2A1            101
A1             102
0              103
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Before leaving cells for the time being, I should point out that actually the full W-graph of a block, as computed by Atlas, has some additional data. First of all, we can (and Atlas does) attach to each edge $i \to j$ a *multiplicity* $m(i,j)$ corresponding to the multiplicity of π_j in $\pi_i \otimes \mathfrak{g}$. Secondly, we can (and Atlas does) attach to each vertex i, the *tau-invariant* of π_i. This is actually an invariant of the annihilator of π_i in $U(\mathfrak{g})$ which I will define below.

1.4. **Primitive Ideals.** A primitive ideal in $U(\mathfrak{g})$ is the annihilator of an irreducible $U(\mathfrak{g})$-module. It should be remarked that the distinct representations can certainly have the same primitive ideal. In fact, different classes of representations can have the same primitive ideal. In fact, the classification of primitive ideals utilizes an entirely different class of representations than the Harish-Chandra modules discussed above.

Let $Prim_\rho$ denote the (finite) set of primitive ideals of infinitesimal character ρ. And let $L(\lambda)$ denote the irreducible $U(\mathfrak{g})$ module of highest weight $\lambda - \rho$. Then

Theorem 1.2 (Duflo)**.** *The map*

$$\varphi : W \to Prim_\rho : w \to Ann\,(L(w\rho))$$

is surjective.

The fibers of this map are called *left cells* in W. Thus, the classification of primitive ideals of infinitesimal character ρ is in terms of certain subsets of W, the *left cells*.

2. Invariants of Primitive Ideals

In what follows we shall need two particular invariants of a primitive ideal of infinitesimal character ρ; its *tau invariant* and an associated *special representation* of W.

2.1. **Tau invariants.**

Let M_w be the Verma module of highest weight $w\rho - \rho$ (containing $L(w\rho)$ as its maximal quotient). Then M_e corresponds to the Verma module with the trivial representation as its maximal quotient and the other extreme is $I_1 \equiv Ann\,(M_{-1}) = Ann\,(L(-\rho))$ which is the unique minimal primitive ideal of infinitesimal character ρ.

Let Π denote the simple roots of $\mathfrak{g}$. And for any $\alpha \in \Pi$, let

$$I_\alpha \equiv Ann\,(L_{s_\alpha}) = Ann\,(M(-s_\alpha\rho)/M(-\rho))$$

Theorem 2.1. *The primitive ideals I_α, $\alpha \in \Pi$, are all distinct from each other and I_1. Any primitive ideal strictly containing I_1 contains at least one of the I_α.*

Definition 2.2. *The* ***tau invariant*** *of a primitive ideal I containing I_1 is*

$$\{s \in \Pi \mid I_s \subset I\}$$

Remark 2.3. One can think of the tau invariant of the primitive ideal as prescribing in along which directions from I a primitive ideal can be reached from I_1 (thinking of the primitive ideals I_s as being nested around I_1 and $s \in tau\,(I)$ meaning $I_1 \subset I_s \subset I$; putting I on the s-side of I_1).

2.2. **Special Representations of W.**

Theorem 2.4 (Joseph)**.** *Let $\mu \in \mathfrak{h}^*$ and consider the function*

$$p : \mathfrak{h}^* \to \mathbb{N} : \mu \to GoldieRank\,(U(\mathfrak{g})/L(\mu))$$

Then for $w \in W$, $p_w : \mu \to p(w\mu)$ is a polynomial on the dominant Weyl chamber. In fact, p_w is a homogeneous harmonic polynomial.

Fact 2.5. *If w and w' belong to the same left cell (i.e, $Ann\,(L_w) = Ann\,(L_{w'})$) then $p_w = p_{w'}$.*

Since p_w is a homogeneous harmonic polynomial on $\mathfrak{h}^*$, W acts on it and thereby generates an irreducible representation of W.

Definition 2.6. *For $\sigma \in \widehat{W}$*

$$\mathcal{C}_\sigma = \{w \in W \mid span_{\mathbb{C}}\,(W \cdot p_w) \approx \sigma\}$$

A representation of $\sigma \in W$ that can be constructed in this fashion is called ***special representation*** *of W. The subset $\mathcal{C}_\sigma \subset W$ is called the* ***two-sided cell*** *in W corresponding to the special representation σ.*

Theorem 2.7.

$$|Prim_\rho| = \sum_{\sigma \in \widehat{W}_{special}} \dim(\sigma)$$

Theorem 2.8. *Suppose $w, w' \in C_\sigma$, then*

$$AV\left(Ann\left(L_w\right)\right) = AV\left(Ann\left(L_{w'}\right)\right)$$

Definition 2.9. *The nilpotent orbits that arise in this fashion are called* ***special*** *nilpotent orbits.*

Remark 2.10. For classical groups, special orbits can be defined as the image of Spaltenstein's order reversing "duality" mapping (based on sending a partition to its conjugate partition). Special orbits in this sense are exactly the special orbits as defined above. Even more succinctly, one can say that an orbit is special if and only it is the associated variety of an irreducible represenation of infinitesimal character ρ.

3. The picture thus far

$$\begin{array}{ccccccc}
 & \text{block } B \subset HC_\rho & & \text{(same inf. char)} & & L_w\ ,\ w \in W & \\
\swarrow & \downarrow & \searrow & & \swarrow & \downarrow & \searrow \\
 & \text{cell } C \subset B & & \text{(same assoc. variety)} & & \mathcal{C}_\sigma\ ,\ \sigma \in \widehat{W}_{special} & \\
\swarrow & \downarrow & \searrow & & \swarrow & \downarrow & \searrow \\
 & ??? & & \text{(same primitive ideal)} & & v \in V_\sigma &
\end{array}$$

4. A partitioning of cells

Recall that the W-graph of a block induces the following graph on a cell: for each element $i \in C$ we attach

- a vertex $v[i]$
- a tau invariant $\tau\left[i\right] =$ tau invariant of $Ann\left(\pi_i\right)$
- a list of edges with multiplicities $e\left[i\right] = \left[\left(j_1, m_1\right), \left(j_2, m_d\right), \ldots, \left(j_k, m_k\right)\right]$

Since a necessary condition for $Ann\left(\pi_i\right) = Ann\left(\pi_j\right)$ it makes sense to group together those cell elements whose vertices have the same tau invariants. Thus, we define a partitioning P_1 of C by grouping together those vertices with the same tau-invariant. Call such a collection a P_1-*subcell* of C.

Next for element i in any particular P_1-subcell, attach the following *second order tau invariant*

$$\tau_{2[i]} = \left\{\tau\left[j\right] \mid j \in \text{edge vertices of } i\right\}$$

and say that two elements i, j belong to the same P_2-*subcell* if

$$\tau_2\left[i\right] = \tau_2\left[j\right] \quad .$$

Similarly, set

$$\tau_3\left[i\right] = \left\{\tau_2\left[j\right] \mid j \in \text{ edge vertices of } i\right\}$$

and say that two elements i, j belong to the sam P_3-subcell if

$$\tau_3\left[i\right] = \tau_3\left[j\right] \quad .$$

Clearly one can continue in this fashion, but eventually since there are only a finite number of cell elements this partitioning must stabilize. Let P_∞ denote the final stable partition (the first P_j for which $P_{j+1} = P_j$).

Theorem 4.1. *Let C be any cell in any real form of any exceptional group G. Then the number of P_∞ subcells that occurs in C is exactly the dimension of a special representation of W.*

Is this a coincidence? Hardly. First of all, it follows from a theorem of Monty McGovern that this partitioning of cells is compatible with a partitioning by collecting together elements with the same primitive ideal. That is,

$$Ann\left(\pi_i\right) = Ann\left(\pi_j\right) \quad \Longrightarrow \quad i, j \text{ belong to same } P_\infty\text{-subcell}$$

The fact that you get exactly the right number of primitive ideals affirms that you are getting **exactly** the the partitioning of the cell by primitive ideals. That is,

Theorem 4.2. *Let C be any cell in any real form of any exceptional group G. Then*

$$i,j \text{ belong to same } P_\infty\text{-subcell in } C \quad \Longrightarrow \quad Ann\left(\pi_i\right) = Ann\left(\pi_j\right)$$

Moreover, a case-by-case analysis also reveals:

Theorem 4.3. *Let B be the big block (maximal split $\times$ maximal split) of an exceptional group E. For each cell C in B construct the set*

$$\tau\left(C\right) = \{\tau\left[i\right] \mid i \in C\}$$

Then

$$\#\left\{\tau\left(C\right) \mid C \in B\right\} = \# \text{ special nilpotent orbits}$$

In other words, the set of tau-invariants of a cell completely characterizes the corresponding associated variety:

$$\tau\left(C\right) = \tau\left(C'\right) \quad \Longleftrightarrow \quad AV\left(\pi_i\right) = AV\left(\pi_j\right) \text{ for any } i \in C \text{ and any } j \in C'$$

Morever, one has the following commutative diagram

$$\begin{array}{ccccc} & & \sigma \in \widehat{W}_{special} & & \\ & \swarrow\nearrow & & \nwarrow\searrow S & \\ \{\tau\left(C\right) \mid C \in B\} & \cdots & \longleftrightarrow & \cdots & \{\text{special nilpotent orbits}\} \end{array}$$

where the maps S is the famous Springer correspondence.

Remark 4.4. By explicitly analyzing of the sets $\{\{\tau\left(C\right) \mid C \in B\}\}$ one can can even identify the Bala-Cartan type of the corresponding orbit.