Tau signatures, orbits and cells, I

O.S.U. Lie Groups Seminar

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1. Introduction

As in my talks last September, the basic setting will be that of a real reductive algebraic group $G_{\mathbb{R}}$ occuring as the set of real points of a linear complex algebraic group $G_{\mathbb{C}}$ defined over $\mathbb{R}$. Again, we'll let $\mathcal{HC}_\lambda$ denote the set of Harish-Chandra modules of irreducible admissible representations of infinitesimal character λ. $\mathcal{HC}_\lambda$ is a finite set of representations parameterized by a certain discrete set $\mathcal{L}_\lambda$ of *Langlands parameters*. The talks I gave in September were about how to figure out when two representations in $\mathcal{HC}_\lambda$ share the same primitive ideal in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} = Lie(G_{\mathbb{R}})_{\mathbb{C}}$; i.e.,

$$\text{for which } x, y \in \mathcal{L}_\lambda \text{ does } Ann\,(\pi_x) = Ann\,(\pi_y)\ ?$$

The basic idea then was to enhance the discrete topology of the Langlands parameter set $\mathcal{L}_\lambda$ with a certain weighted graph structure and then observe that for $Ann\,(\pi_x) = Ann\,(\pi_y)$ it was necessary and sufficient that the local graph of x be identical to the local graph of y. (Albeit, that is not quite how I put it in September.)

What I want to discuss about in the next couple of talks is how one can explicitly identify the associated variety of the annihilator $Ann(\pi_x)$ of a representation $\pi_x \in \mathcal{HC}_\lambda$.

It might be worthwhile to present a diagram that sketches the distinction between the two notions of associated varieties and the two common routes from representations to nilpotent orbits in $\mathfrak{g}$:

$$\begin{array}{ccccc} & & J_x = Ann\,(\pi_x) \in Prim_\lambda & & \\ & \nearrow & & \searrow gr & \\ \mathcal{HC}_\lambda \ni \pi_x & & & & AV(J_x) = \overline{\mathcal{O}_x} \in G\backslash\mathcal{N}_{\mathfrak{g}} \\ & \searrow gr & & G_{\mathbb{C}} \nearrow & \\ & & AV\,(\pi_x) \subset K_{\mathbb{C}}\backslash(\mathfrak{g}/\mathfrak{k})^* & & \end{array}$$

We note these maps are all surjective, and appearing as it does at the tail end of a sequence of surjections, the associated variety $\overline{\mathcal{O}_x}$ of the annihilator of a representation π_x is relatively weak invariant. But weaker invariants can in fact be better invariants. For the weaker the invariant, the broader the reach of the corresponding notions of similarity and proximity. (I myself think of this as a sort of *conceptual uncertainity principle*.) In any case, the associated variety of the annihilator of a representation is important because it attaches to a representation an algebraic geometric object amenable to algebraic analysis - where even such mundane notions as dimension pull back to deeper abstract properties on the pure representation-theoretical side of things.

Today's talk will be focused on the associated varieties of the annihilators $Ann(\pi_x) \subset U(\mathfrak{g})$ of irreducible admissible representations $\pi_x \in \mathcal{H}_\lambda$ of regular integral infinitesimal character. Each of these is, as is well-known, the (Zariski) closure of a single nilpotent orbit $\mathcal{O}_x$ of the complexified adjoint group $G = Ad(\mathfrak{g})$ in the nilpotent cone $\mathcal{N} = \mathcal{N}_{\mathfrak{g}}$ of $\mathfrak{g}$.[1] In fact, arising as they do from irreducible representation of *regular integral infinitesimal character*, the orbits we consider today will be what-are-known-as *special nilpotent orbits*. My goal for today is to simply to bring to light a particular attribute, the *tau signature*, of a *special* nilpotent orbit $\mathcal{O}$ Next time, we'll use this signature to match irreducible admissible representations of regular integral infinitesimal character to the associated varieties of their annihilators.

[1]Throughout this talk we'll be making use of an identification $\mathfrak{g} \approx \mathfrak{g}^*$ arising, say, from some non-degenerate invariant bilinear form on $\mathfrak{g}$. $\mathcal{N}_{\mathfrak{g}}$ will denote the cone of nilpotent elements in $\mathfrak{g}$. Occasionally, when the identity of the underlying Lie algebra is clear, we'll simply write $\mathcal{N}$ for $\mathcal{N}_{\mathfrak{g}}$.

2. Spaltenstein-Barbasch-Vogan duality

Let me begin with the prototypical example.

Let $\mathfrak{g} \approx A_{n-1} \approx \mathfrak{sl}_n(\mathbb{C})$. As is well know, the nilpotent orbits in $\mathfrak{g}$ can be parameterized by partitions of n. Explicitly, if

$$\mathbf{p} = [p_1, \ldots, p_k] \quad , \quad p_1 \geq p_2 \geq \cdots \geq p_k > 0 \quad , \quad \sum_{i=1}^{k} p_k = n$$

is a partition of n, we can set

$$\mathcal{O}_{\mathbf{p}} = \left\{ g \begin{pmatrix} \mathbf{J}_{p_1} & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{p_2} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \mathbf{J}_{p_{k-1}} & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{J}_{p_k} \end{pmatrix} g^{-1} \mid g \in SL_n(\mathbb{C}) \right\} ,$$

where each $\mathbf{J}_\ell$ is an $\ell \times \ell$ Jordan block of the form

$$\mathbf{J}_\ell = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 \\ 0 & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

(N.B., $\mathbf{J}_1 = [0]$ and so the partition $[1, \ldots, 1]$ corresponds to the orbit of the zero matrix.) The correspondence

$$\mathcal{P}_n \ni \mathbf{p} \longrightarrow \mathcal{O}_{\mathbf{p}} \in G \backslash \mathcal{N}$$

is a bijection. (See e.g., §3.1 and Chapter 5 in Collingwood-McGovern.)

Now what's especially nice about this parameterization is that the set of orbits inherits many of the structural properties of the set $\mathcal{P}_n$ of partitions of n.

For example, there is a canonical partial ordering of partitions - the so-called *dominance ordering* defined by

$$\mathbf{p} \leq \mathbf{p}' \quad \Longleftrightarrow \quad \text{for each } 1 \leq i \leq n \quad \sum_{j=1}^{i} p_j \leq \sum_{j=1}^{i} p'_j$$

Fact 2.1. *Let $\mathbf{p}, \mathbf{p}' \in \mathcal{P}_n$ and let $\mathcal{O}_{\mathbf{p}}$, $\mathcal{O}_{\mathbf{p}'}$ be the corresponding nilpotent orbits for $SL_n(\mathbb{C})$, then*

$$\mathbf{p} \leq \mathbf{p}' \quad \Longleftrightarrow \quad \mathcal{O}_{\mathbf{p}} \subseteq \mathcal{O}_{\mathbf{p}'} \quad .$$

There is also a canonical order-reversing involution on $\mathcal{P}_n$: the *transpose map*

$$d : \mathcal{P}_n \to \mathcal{P}_n : d(\mathbf{p}) = \mathbf{p}^t$$

where if $\mathbf{p} = [p_1, \ldots, p_k]$

$$\left(\mathbf{p}^t\right)_i = \#\{j \text{ such that } p_j \geq i\}$$

This involution of $\mathcal{P}_n$ induces a involution on the set of nilpotent orbits

$$d : G\backslash\mathcal{N}_{\mathfrak{g}} \to G\backslash\mathcal{N}_{\mathfrak{g}} \ , \ \mathcal{O}_{\mathbf{p}} \mapsto \mathcal{O}_{\mathbf{p}^t}$$

which we shall refer to as the *Spaltenstein duality* map.

There is similar but slightly more complicated Spaltenstein duality map for the other simple Lie algebras. Before describing it, perhaps we should first recall the partition-parameterization of nilpotent orbits for the other classical complex Lie algebras.

Fact 2.2.
- *The nilpotent orbits of simple Lie algebras of type B_n are parameterized the set $\mathcal{P}_{B_n}$ by partitions of $2n+1$ for which the even parts occur with even multiplicity;*
- *The nilpotent orbits of simple Lie algebras of type C_n are parameterized by the set $\mathcal{P}_{C_n}$ of partitions of $2n$ for which the odd parts occur with even multiplicity;*
- *The nilpotent orbits of simple Lie algebras of type D_n are (nearly) parameterized by the set $\mathcal{P}_{D_n}$ of partitions of $2n$ for which the even parts occur with even multiplicity. If a partition is* very even, *meaning only even parts occur and these parts have even multiplicities, an additional label I, or II is needed to separate orbits.*

Notation 2.3. So that we can treat the classical groups uniformly, we'll let $\mathcal{P}_G$ denote the set of partitions parameterizing a classical group G (so, in particular, when $G \approx A_n$, $\mathcal{P}_G = \mathcal{P}_{n+1}$.)

Fact 2.1 generalizes to the following statement:

Fact 2.4. *Let G be a classical group, $\mathbf{p}, \mathbf{p}' \in \mathcal{P}_G$ and let $\mathcal{O}_{\mathbf{p}}$, $\mathcal{O}_{\mathbf{p}'}$ be the corresponding nilpotent orbits for G, then*

$$\mathbf{p} \leq \mathbf{p}' \quad \Longleftrightarrow \quad \mathcal{O}_{\mathbf{p}} \subseteq \mathcal{O}_{\mathbf{p}'} \quad .$$

Fact 2.2, however, does not extend so easily. For, unless $G \approx SL_n(\mathbb{C})$, the transpose map $\mathbf{p} \mapsto \mathbf{p}^t$ is not an involution of $\mathcal{P}_G$; indeed $\mathbf{p}^t$ need not lie in $\mathcal{P}_G$. However, it is true that there is a unique maximal partition $(\mathbf{p}^{\mathbf{t}})_G \in \mathcal{P}_G$ such that $(\mathbf{p}^{\mathbf{t}})_G \leq \mathbf{p}^t$ (inside $\mathcal{P}_n$). $(\mathbf{p}^{\mathbf{t}})_G$ is called the *G-collapse* of $\mathbf{p}^t$ and through it one can define a Spaltenstein duality mapping

$$d : G\backslash\mathcal{N} \to G\backslash\mathcal{N} \quad : \quad \mathcal{O}_{\mathbf{p}} \mapsto \mathcal{O}_{(\mathbf{p}^{\mathbf{t}})_G}$$

Fact 2.5. *If $\mathcal{O} \subseteq \overline{\mathcal{O}'}$ then $d(\mathcal{O}') \subseteq \overline{d(\mathcal{O})}$.*

Fact 2.6. *Although the Spaltenstein duality map is not an involution on $G\backslash\mathcal{N}$, when restricted to its range it is an involution:*

$$d \circ d \circ d = d \ .$$

Definition 2.7. *If $\mathcal{O} \subset \mathcal{N}$ lies in the image of the Spaltenstein duality map, we shall say that $\mathcal{O}$ is a* ***special*** *nilpotent orbit.*

Fact 2.8. *There are several other equivalent characterizations of special nilpotent orbits (valid also in the context of exceptional groups) that will be useful later on:*

- *A nilpotent orbit $\mathcal{O}$ is special if its closure is the associated variety of a primitive ideal of regular integral infinitesimal character.*
- *A nilpotent orbit $\mathcal{O}$ is special if its associated Weyl group representation (via the Springer correspondence) is a special representation of the Weyl group. (The point being that there is an intrinsic characterization of a* ***special Weyl group representation*** *and the Springer correspondence preserves "special-ness".)*

As alluded to above, there is also a notion of a Spaltenstein duality map and a notion of special orbits for the exceptional groups. However, since the nilpotent orbits of the special groups have little to do with partitions, the original construction of the Spaltenstein duality map for the exceptional groups was a bit *ad hoc*. However, Barbasch and Vogan in their paper, *Unipotent representations of complex semisimple groups*, provided a uniform construction of a duality map

$$\eta : G\backslash\mathcal{N}_{\mathfrak{g}} \to G^\vee\backslash\mathcal{N}_{\mathfrak{g}^\vee}$$

mapping nilpotent orbits of G to nilpotent orbits in its corresponding dual group. When $\mathfrak{g}$ is simply-laced (so that $\mathfrak{g} \approx \mathfrak{g}^\vee$) the map η coincides with the Spaltenstein duality map, and even when $\mathfrak{g} \not\approx \mathfrak{g}^\vee$, it can be interpreted as leading to the same Spaltenstein duality map (e.g. by setting $d = \sigma_G^{-1} \circ \sigma_{G^\vee} \circ \eta$, where $\sigma_G : G \backslash \mathcal{N}_{\mathfrak{g}} \longleftrightarrow \widehat{W_G}$ is the Springer correspondence).

3. Levi subalgebras and nilpotent orbits

In this section, $\mathfrak{g}$ will be a complex reductive Lie algebra and $G = Ad(\mathfrak{g})$ will be the (complex) adjoint group of $\mathfrak{g}$. We recall the following definitions.

Definition 3.1. *Let $\mathfrak{g}$ be a complex reductive Lie algebra. A* Borel subalgebra *of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$, a* parabolic subalgebra *of $\mathfrak{g}$ is a subalgebra containing a Borel subalgebra, and a* Levi subalgebra *of $\mathfrak{g}$ is the reductive part of a parabolic subalgebra (a parabolic algebra $\mathfrak{p}$ can always be written as a semidirect sum $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$, with $\mathfrak{l}$ reductive and $\mathfrak{n}$ nilpotent).*

There are two natural ways of constructing nilpotent orbits in $\mathfrak{g}$ from nilpotent orbits of Levi subalgebras: Bala-Carter inclusion and (parabolic) induction:

3.1. **Bala-Carter inclusion.** Let $\mathfrak{l}$ be a Levi subalgebra of $\mathfrak{g}$ and let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in $\mathfrak{l}$. Then

$$inc_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = G \cdot \mathcal{O}_{\mathfrak{l}}$$

will be a nilpotent orbit in $\mathfrak{g}$.

Remark 3.2. One says that a nilpotent orbit $\mathcal{O}$ is *distinguished* if no $X \in \mathcal{O}$ is contained in a proper Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$. The Bala-Carter parameterization of nilpotent orbits amounts to generating, in a one-to-one fashion, all the nilpotent orbits of $\mathfrak{g}$ from (conjugacy classes of) distinguished orbits of Levi subalgebras via the inclusion method.

3.2. **Parabolic induction.** The second method of constructing a nilpotent orbit of $\mathfrak{g}$ from a nilpotent orbit of a Levi subalgebra is by *parabolic induction.* Here one starts with an orbit $\mathcal{O}_{\mathfrak{l}}$ of a Levi subalgebra $\mathfrak{l}$, chooses a parabolic $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ wherein $\mathfrak{l}$ appears as the Levi factor ($\mathfrak{p}$ is not unique), and then sets

$$ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = \text{unique dense orbit in } G \cdot (\mathcal{O}_{\mathfrak{l}} + \mathfrak{n})$$

It is a fact that the resulting orbit does not depend on the choice of $\mathfrak{p}$. Nevertheless, this method is commonly referred to as parabolic induction of orbits because of its close connection with parabolic induction of representations.

3.3. **Inclusion, induction and duality.** The following result is what David Vogan [Vo] refers to as a "serious theorem" in [BV]

Theorem 3.3. *Suppose $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are dual Lie algebras with dual Levis $\mathfrak{l}$ and $\mathfrak{l}^\vee$. Duality, Bala-Carter inclusion, and parabolic induction satisfy*

$$\eta_{\mathfrak{g}^\vee}\left(inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee}(\mathcal{O}_{\mathfrak{l}}^\vee)\right) = ind_{\mathfrak{l}}^{\mathfrak{g}}(\eta_{\mathfrak{l}^\vee}(\mathcal{O}_{\mathfrak{l}}^\vee))$$

3.4. **Principal includes, Richardson orbits, and a special property of special orbits.** Fix a Levi subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$. Obviously the ordering of orbits by inclusion in closures is preserved by Bala-Carter inclusion: i.e., if $\mathcal{O}_{\mathfrak{l}}' \subseteq \overline{\mathcal{O}_{\mathfrak{l}}} \subset \mathcal{N}_{\mathfrak{l}}$, then

$$inc_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}') \subseteq \overline{inc_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})}$$

Thus, the largest orbit in $\mathcal{N}_{\mathfrak{g}}$ that is obtainable from an orbit in $\mathcal{N}_{\mathfrak{l}}$ by Bala-Carter inclusion is $inc_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l},prin})$, where $\mathcal{O}_{\mathfrak{l},prin}$ is the principal nilpotent orbit in $\mathcal{N}_{\mathfrak{l}}$. Set

$$\mathcal{I} = \left\{ inc_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l},prin}) \mid \mathfrak{l} \text{ a Levi subalgebra of } \mathfrak{g} \right\}$$

Lacking of a standard nomenclature, we'll simply refer to the orbits in $\mathcal{I}$ as *principal includes.*

Among the induced orbits attached to a Levi subalgebra there orbit induced from the principal orbit will also be maximal. However, we shall be more interested in the minimal induced orbits; that is, the orbits induced from the trivial orbits of Levi subalgebras. These are called *Richardson orbits.* We let $\mathcal{R}$ denote the set of Richardson orbits

$$\mathcal{R} = \left\{ ind_{\mathfrak{l}}^{\mathfrak{g}} \left(\mathbf{0}_{\mathfrak{l}} \right) \mid \mathfrak{l} \text{ a Levi subalgebra of } \mathfrak{g} \right\}$$

Remark 3.4. Since induction preserves "special-ness" and because the trivial orbit is always special, Richardson orbits are always special orbits.

Fact 3.5. *Let $\mathcal{O}$ be a special nilpotent orbit in a complex simple classical algebra. Set*

$$\mathcal{I}(\mathcal{O}) = \left\{ \mathcal{O}' \in \mathcal{I} \mid \mathcal{O}' \subseteq \overline{\mathcal{O}} \right\}$$
$$\mathcal{R}(\mathcal{O}) = \left\{ \mathcal{O}' \in \mathcal{R} \mid \mathcal{O} \subseteq \overline{\mathcal{O}'} \right\}$$

Then $\mathcal{O}$ is completely determined by the sets $\mathcal{I}(\mathcal{O})$ and $\mathcal{R}(\mathcal{O})$.

Since the Barbasch-Vogan duality is ordering reversing

$$\eta_{\mathfrak{g}} \left(\mathcal{O}_{\mathfrak{g},prin} \right) = \mathbf{0}_{\mathfrak{g}^\vee} \quad \text{and} \quad \eta_{\mathfrak{g}} \left(\mathbf{0}_{\mathfrak{g}} \right) = \mathcal{O}_{\mathfrak{g}^\vee,prin}$$

Thus, by the Barbasch-Vogan theorem

$$\eta_{\mathfrak{g}^\vee} \left(inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee} \left(\mathcal{O}_{\mathfrak{l}^\vee,prin} \right) \right) = ind_{\mathfrak{l}}^{\mathfrak{g}} \left(\eta_{\mathfrak{l}^\vee} \left(\mathcal{O}_{\mathfrak{l}^\vee,prin} \right) \right) = ind_{\mathfrak{l}}^{\mathfrak{g}} \left(\mathbf{0}_{\mathfrak{l}} \right)$$

Applying the order-reversing property of the the Barbasch-Vogan duality map again, we see that

$$inc_{\mathfrak{l}^\vee}^{\mathfrak{g}^\vee} \left(\mathcal{O}_{\mathfrak{l}^\vee,prin} \right) \subset \overline{\mathcal{O}} \quad \Longrightarrow \quad \eta_{\mathfrak{g}} \left(\mathcal{O} \right) \subseteq ind_{\mathfrak{l}^\vee}^{\mathfrak{g}} \left(\mathbf{0}_{\mathfrak{l}^\vee} \right)$$

and so

$$\mathcal{I}(\mathcal{O}) \longleftrightarrow \mathcal{R}(\eta(\mathcal{O}))$$

Thus, the above fact can be re-stated as

Fact 3.6. *Every special orbit $\mathcal{O}$ is specified by knowing the sets $\mathcal{R}(\mathcal{O})$ and $\mathcal{R}(\eta(\mathcal{O}))$.*

4. The tau signature of an orbit

We'll soon conclude this talk with yet another consequence or recharacterization of Fact 3.7.

First of all, the sets $\mathcal{R}(\mathcal{O})$ are determined by their minimal elements

$$\mathcal{R}_{\min}(\mathcal{O}) = \{ \mathcal{O}' \in \mathcal{R}(\mathcal{O}) \mid \mathcal{O}' \text{ is not contained in the closure of any other } \mathcal{O}'' \in \mathcal{R}(\mathcal{O}) \}$$

Secondly, for any subset Γ of the set of simple roots $\Pi \subset \Delta(\mathfrak{h}, \mathfrak{g})$, there is a corresponding *standard Levi subsubalgebra* $\mathfrak{l}_\Gamma$, such that $[\mathfrak{l}_\Gamma, \mathfrak{l}_\Gamma]$ is generated by the simple roots in Γ, and the association

$$\Gamma \mapsto \mathfrak{l}_\Gamma$$

hits every conjugacy class of Levi subalgebras as Γ ranges over the power set of Π. This association is far from 1:1, however. What is *almost* true is that if $\mathfrak{l}_\Gamma \approx \mathfrak{l}_{\Gamma'}$ as Lie algebras then $\mathfrak{l}_\Gamma$ and $\mathfrak{l}_{\Gamma'}$ are conjugate under $Ad(\mathfrak{g})$ and so produce the same Richardson orbit. The exceptions to this situation are

- For types B_n and C_n there are two conjugacy classes of rank 1 Levi subalgebra (corresponding to $\{\alpha\} = \Gamma$ being a long or short root).
- For types D_n the outer-automorphism $\alpha_{n-1} \longleftrightarrow \alpha_n$ leads to some non-conjugate but isomorphic Levis
- For E_7 there are three pairs of isomorphic but non-conjugate Levi subalgebras (where $\mathfrak{l} \approx 3A_1,\ A_3 + A_1$, or A_5)

Definition 4.1. *A **set of standard Gammas** is a collection $\Psi = \{\Gamma_i \mid i \in I\}$ of subsets of Π such that the association*

$$\Gamma \in \Psi \to Ad(\mathfrak{g})\text{-conjugacy class of } \mathfrak{l}_\Gamma$$

is a bijection.

Next time I'll describe a easy, natural way to set up a set of standard gammas for the classical groups which will allow an easy determination of the partitions corresponding to corresponding Richardson orbits. Today, however, I'll simply conclude with the following recharacterization of Fact 3.7:

Fact 4.2. *Fix a special orbit $\mathcal{O}$, suppose*

$$\mathcal{R}_{\min}(\mathcal{O}) = \left\{ ind_{\mathfrak{l}_{\Gamma_1}}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}_{\Gamma_1}}\right), \ldots, ind_{\mathfrak{l}_{\Gamma_k}}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}_{\Gamma_k}}\right)\right\}$$
$$R_{\min}(\eta(\mathcal{O})) = \left\{ ind_{\mathfrak{l}_{\Phi_1}}^{\mathfrak{g}^\vee}\left(\mathbf{0}_{\mathfrak{l}_{\Phi_1}}\right), \ldots, ind_{\mathfrak{l}_{\Phi_\ell}}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}_{\Phi_\ell}}\right)\right\}$$

and set

$$\tau(\mathcal{O}) = \{\Gamma_1, \ldots, \Gamma_k\} \subset 2^{\Pi}$$
$$\tau^\vee(\mathcal{O}) = \{\Phi_1, \ldots, \Phi_\ell\} \subset 2^{\Pi^\vee}$$

Then, $\mathcal{O}$ is completely determined by the pair $(\tau(\mathcal{O}), \tau^\vee(\mathcal{O}))$.

Definition 4.3. *Let $\mathcal{O}$ be a special orbit for a complex simple Lie algebra, and let $\tau(\mathcal{O})$ and $\tau^\vee(\mathcal{O})$ be as above. We call the pair $(\tau(\mathcal{O}), \tau^\vee(\mathcal{O}))$ the **tau signature** of $\mathcal{O}$.*

Remark 4.4. As the name implies, tau signatures will have something to do with tau invariants. This connection we shall make next time.

References

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[Vo] D. Vogan (Atlas Wiki entry, August 2007).