# Tau signatures, orbits and cells, II

O.S.U. Lie Groups Seminar

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## 1. Recap

Let me begin by recalling the general setup and principal objects.

**The back-story:**

- $G_{\mathbb{R}}$ : a real reductive Lie group realizable as the set of real points of a complex linear algebraic group defined over $\mathbb{R}$;
- $\mathcal{L}$ : set of Langlands parameters for $G_{\mathbb{R}}$;
- $\mathcal{L}_{\lambda}$ : set of Langlands parameters for irreducible admissible representations of regular integral infinitesimal character $\lambda$;
- $\widehat{G_{\mathbb{R},\lambda}} = \{\pi_x \mid x \in \mathcal{L}_{\lambda}\}$ : admissible representations of regular integral infinitesimal character $\lambda \in \mathfrak{h}^*$;
- $\mathcal{HC}_{\lambda} = \left\{V_x = \pi_x|_{K-finite} \mid x \in \mathcal{L}_{\lambda}\right\}$: set of irreducible Harish-Chandra ($(\mathfrak{g}, K)$-) modules corresponding to irreducible admissible representations $\pi_x \in \widehat{G_{\mathbb{R}}}$, $x \in \mathcal{L}_{\lambda}$.

**In the last episode:**

- $\mathfrak{g} = Lie\,(G_{\mathbb{R}})_{\mathbb{C}}$; $\mathfrak{h}$, a CSA for $\mathfrak{g}$; $\Delta = \Delta(\mathfrak{h}, \mathfrak{g})$, roots of $\mathfrak{h}$ in $\mathfrak{g}$; $\Pi \subset \Delta$, choice of simple roots in $\Delta$, $\Delta^+ = \Delta^+(\mathfrak{h}, \mathfrak{g}; \Pi)$ set of positive roots.
- $G$ : adjoint group of $\mathfrak{g} = Lie\,(G_{\mathbb{R}})_{\mathbb{C}}$
- $Prim_{\lambda} = \{J_x \in Prim\,(U(\mathfrak{g})) \mid J_x = Ann\,(V_x)\ ,\ x \in \mathcal{L}_{\lambda}\}$
- $\mathcal{N}_{\mathfrak{g}}$ : nilpotent cone in $\mathfrak{g}$
- $G\backslash\mathcal{N}_{\mathfrak{g}}$ : nilpotent orbits in $\mathfrak{g}$
- $\mathcal{S} = \{AV\,(J_x) \mid x \in \mathcal{L}_{\lambda}\}$ : the special nilpotent orbits in $G\backslash\mathcal{N}_{\mathfrak{g}}$
- $d : G\backslash\mathcal{N}_{\mathfrak{g}} \to \mathcal{S}$ : the SBV-duality map that restricts to an involution on $\mathcal{S}$.
- $\Gamma$ : a subset of the simple roots.
- $\mathfrak{l}_{\Gamma}$ : standard Levi subalgebra attached to $\Gamma \subset \Pi$;

$$\mathfrak{l}_{\Gamma} = \mathfrak{h} + \sum_{\alpha \in \langle\Gamma\rangle} \mathfrak{g}_{\alpha}$$

- $R_{\Gamma} = ind_{\mathfrak{l}_{\Gamma}}^{\mathfrak{g}}\,(\mathbf{0}_{\mathfrak{l}_{\Gamma}})$ : the Richardson orbit induced from the trivial orbit of a Levi subalgebra $\mathfrak{l}_{\Gamma}$ of $\mathfrak{g}$
- **Fact:** every special nilpotent orbit $\mathcal{O} \in \mathcal{S}$ is determined by the following closure data

$$\mathcal{R}_{\min}(\mathcal{O}) = \{\text{smallest Richardson orbits whose closures contain } \mathcal{O}\ \}$$
$$\mathcal{R}^{\vee}_{\min}(\mathcal{O}) = \{\text{smallest Richardson orbits whose closures contain } d(\mathcal{O})\}$$

- $\Psi = \{\Gamma \subset \Pi\}$ : a set of *standard Gammas*: a collection of $\Gamma \in 2^{\Pi}$ such that

$$i : \Psi \to \{\text{conjugacy classes of Levi subalebras of } \mathfrak{g}\} \quad , \quad i(\Gamma) = Ad\,(G)\,\mathfrak{l}_{\Gamma}$$

  is a bijection.
- $(\tau(\mathcal{O}), \tau^{\vee}(\mathcal{O}))$ : the *tau signature* of a special orbit, where

$$\tau(\mathcal{O}) = \left\{\Gamma \in \Psi \mid ind_{\mathfrak{l}_{\Gamma}}^{\mathfrak{g}}\,(\mathbf{0}_{\mathfrak{l}_{\Gamma}}) \in \mathcal{R}_{\min}(\mathcal{O})\right\}$$
$$\tau^{\vee}(\mathcal{O}) = \left\{\Gamma \in \Psi \mid ind_{\mathfrak{l}_{\Gamma}}^{\mathfrak{g}}\,(\mathbf{0}_{\mathfrak{l}_{\Gamma}}) \in \mathcal{R}^{\vee}(\mathcal{O})\right\}$$

## 2. The Plan for Today

- Tau signatures of cells
  - H-C cells
  - The Spaltenstein-Vogan Criterion
  - The cell-orbit correspondence
- Explicit cell-orbit correspondence
  - the exceptional group case
  - the classical cases
    - standard Gammas to Richardson orbits to partitions
    - computing tau signatures of special orbits

## 3. Tau signatures for Harish-Chandra cells

Let me recall the cell decomposition of the set $\mathcal{HC}_\lambda$ of irreducible Harish-Chandra modules of infinitesimal character $\lambda$. We fix a regular integral infinitesimal character $\lambda$, and let $\mathcal{L}_\lambda$ denote the (finite) set of Langlands parameters fof $\mathcal{HC}_\lambda$. For $x \in \mathcal{L}_\lambda$, we write $V_x$ for the corresponding Harish-Chandra module. For any pair $x, y \in \mathcal{L}_\lambda$, we say

$$x \rightsquigarrow y \quad \Longleftrightarrow \quad \exists \text{ a f.d. representation } F \text{ (occuring in } T(\mathfrak{g})\text{) such that } V_y \text{ appears in } V_x \otimes F$$

and that

$$x \sim y \quad \Longleftrightarrow \quad x \rightsquigarrow y \text{ and } y \rightsquigarrow x$$

Then $\sim$ is an equivalence relation on $\mathcal{HC}_\lambda$ and the corresponding equivalence classes are called Harish-Chandra cells.

**Fact 3.1.** *If $x, y$ belong to the same cell, then*

$$AV(V_x) = AV(V_y) \quad \Longrightarrow \quad AV(Ann(V_x)) = AV(Ann(V_y))$$

*Here $AV(V_x)$ is the associated variety of the $(\mathfrak{g}, K)$-module $V_x$, a collection of $K_{\mathbb{C}}$-orbits in $(\mathfrak{g}/\mathfrak{k})^*$ and $AV(Ann(V_x))$ is the associated variety of the primitive ideal corresponding to $x \in \mathcal{L}_\lambda$, the closure of a single $G_{\mathbb{C}}$-orbit in $\mathfrak{g}$. Both of these equivalences follow from the circumstance that tensoring by a finite-dimensional representation doesn't really affect the outcome of the gradation processes by which the associated varieties are constructed.*

*Notation* 3.2. For any cell $\mathcal{C} \subset \mathcal{L}_\lambda$, let $\mathcal{O}_{\mathcal{C}}$ be the orbit in $\mathcal{N}_{\mathfrak{g}}$ whose closure is the common associated variety of the annihilators of the corresponding set of representations.

Another, equivalent characterization of Harish-Chandra cells goes like this. Identify the set $\mathcal{L}_\lambda$ as set of vertices of a weighted directed graph $\mathcal{G}$ where the *weight* of a vertex $x \in \mathcal{G}$ is the tau-invariant $\tau(x) \subset \Pi$ of the primitive ideal corresponding to $V_x$ and for which there is an *edge* $x \to y$ from $x$ to $y$ whenever $V_y$ appears in $V_x \otimes \mathfrak{g}$. The Harish-Chandra cells then correspond to the subgraphs in which every ordered pair $(x, y)$ of vertices can be connected by a directed path (along the directed edges) from $x$ to $y$. So in particular, if $x, y$ belong to the same cell, then there is a directed path from $x$ to $y$ as well as a directed path from $y$ to $x$.

The Atlas software computes these graphs as a by-product of the KLV-polynomial computations. Here is a remarkable empirical fact:

*Observation* 3.3. Let $\lambda$ be a regular and integral. For each cell $\mathcal{C} \in \mathcal{L}_\lambda$, form

$$\tau(\mathcal{C}) = \{\tau(x) \mid x \in \mathcal{C}\}$$

Then, for simple split real groups,

$$\#\{\tau(\mathcal{C}) \mid \mathcal{C} \subset \mathcal{L}_\lambda\} = \# \text{ special nilpotent orbits in } \mathcal{N}_{\mathfrak{g}}$$

Recall that the special nilpotent orbits are exactly the orbits that appear as the associated varieties of primitive ideals of regular integral infinitesimal character. Evidently, the set $\tau(\mathcal{C})$ has something to do with the orbit $\mathcal{O}_{\mathcal{C}}$. Indeed,

**Theorem 3.4** (Vogan)**.** *Suppose $\mathcal{C}$ is a cell of Harish-Chandra modules with associated complex nilpotent orbit $\mathcal{O}_{\mathcal{C}}$. Then $\mathcal{O}_{\mathcal{C}}$ is contained in the closure of $ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}})$ if and only if $\mathcal{C}$ contains an element $x$ whose $\tau$-invariant $\tau(x)$ contains the simple roots of $\mathfrak{l}$.*

*Ideas behind the proof.* In [ Sp1], Spaltenstein proves that a special orbit $\mathcal{O}$ is contained in a Richardson orbit $ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}})$ if and only if the sign representation of $W_{\mathfrak{l}}$ occurs in the restriction of the Springer representation $\sigma(\mathcal{O}) \in \widehat{W}_{special}$ corresponding to $\mathcal{O}$ to $W_{\mathfrak{l}}$. Vogan then observed that $\sigma(\mathcal{O}_{\mathcal{C}})$ must also occur in the coherent continuation representation of $W$ attached to the cell,[1] and that for the sign representation to occur in coherent continuation representation attached to a cell $\mathcal{C}$ it is necessary and sufficient to have at least one $x \in \mathcal{C}$ such that the simple roots of $\mathfrak{l}$ all lie in $\tau(x)$.[2]

Here is another remarkable fact,

**Fact 3.5.** *For each HC cell $C$ in a block there is a corresponding dual cell $C^\vee$ with the property that*

$$\{\tau(y) \mid y \in C^\vee\} = \{\tau_c(x) \mid x \in C\}$$

*where $\tau_c(x)$ is the complement of $\tau(x)$ in $\Pi$.*

**Definition 3.6.** *Let $C$ be an HC cell and set*

$$\tau(C) = \{\tau(x) \mid x \in C\}$$

*We partial order the elements of $\tau(C)$ as follows: each $\tau(x) \in \tau(C)$ is a certain subset of the simple roots, and so corresponds to a certain Levi subalgebra $\mathfrak{l}_{\tau(x)}$ of $\mathfrak{g}$ and hence to a certain Richardson orbit*

$$\mathcal{O}_{\tau(x)} = ind_{\mathfrak{l}_{\tau(x)}}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}_{\tau(x)}}\right)$$

*We say*

$$\tau(x) \leq \tau(y) \quad \Longleftrightarrow \quad \mathcal{O}_{\tau(x)} \subset \overline{\mathcal{O}_{\tau(y)}} \quad ,$$

*set*

$$\tau_{\min}(C) = \{\textit{minimal } \tau(x) \in \tau(C)\}$$

*and define the **tau signature** of a cell as the pair*

$$\tau_{sig}(C) = (\tau_{\min}(C), \tau_{\min}(C^\vee)) \quad .$$

**Corollary 3.7** (B)**.** *A special orbit $\mathcal{O}$ is the nilpotent orbit attached to a cell $C$ if and only if*

$$\tau_{sig}(\mathcal{O}) = \tau_{sig}(C)$$

## 4. Explicit cell-orbit correspondences

As per the Corollary above, to figure out the explicit cell/orbit correspondences we just need to compute the tau signatures of HC cells and the tau signatures of special nilpotent orbits; and then match them up.

Now the the tau-signatures of cells which can be obtained from the output of the Atlas `wcells` command as follows. `wcells` produces a listing of the irreducible representaions in a block, cell-by-cell, along with their tau-invariants and other data connected to the W-graph of the block. All one has to do is observe which tau-invariants appear in which cell and collect the minimal ones (with respect to the ordering of Gammas induced by the ordering of Richardson orbits) into a set $\tau_{min}(C)$, It is easy to figure out the duality map

[1] by consistency with Springer correspondence

[2] In Vogan's "Big Green Book", where the coherent continuation representation is developed, $s_\alpha \cdot \Theta(x) = -\Theta(x)$ is a condition that puts $\alpha \in \tau(x)$

$C \leftrightarrow C^{\vee}$ amongst the cells using Fact 3.5, and then write down the tau signature of a cell as prescribed by Definition 3.6.

As for the tau-signatures of orbits: well, because one normally parameterizes the nilpotent orbits of classical groups using partitions, and uses Bala-Carter data to parameterize the nilpotent orbits of the exceptional groups we have two separate methods to relate tau signatures to nilpotent orbit parameters.

- The tau-signatures of special nilpotent orbits for the exceptional groups can be obtained by consulting Spaltenstein's book. In [ Sp2], one finds not only Hasse diagrams showing the closure relations among the special nilpotent orbits, but also an explicit description of the duality map, and tables listing the induced nilpotent orbits, in particular the Richardson orbits, for the exceptional groups. In short, everything you need to figure out the tau signature of a particular orbit.
- Figuring out tau-signatures of special nilpotent orbits for the classical groups requires a little more thinking. However, once one absorbs Chapters 5, 7, and 8 of Collingwood and McGovern [ CM], it's not too hard to figure out how to compute the partition corresponding to a Richardson orbit corresponding to a given Levi $\mathfrak{l}_\Gamma$:
$$\Gamma \to R_\Gamma = ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}) \longleftrightarrow \mathbf{p}_\Gamma \in \mathcal{P}_G$$
One can then use the dominance partial ordering of partitions to figure out the minimal Richardson orbits $\mathcal{O}_{\mathbf{p}_\Gamma}$ that contain a given special orbit $\mathcal{O}_{\mathbf{p}}$, and then, using an implementation of the Spaltenstein duality map on partitions, the minimal Richardson orbits that contain its Spaltenstein dual. I'll describe the underlying algorithms in a little more detail in the next section.

## 5. Computing tau signatures for the special nilpotent orbits of classical groups

Before getting into the details of the computation, I should probably point out that the basic strategy to be pursued is quite simple. Recall that the nilpotent orbits of a complex classical Lie algebra $\mathfrak{g}$ can be parameterized by certain families $\mathcal{P}_\varepsilon(N_{\mathfrak{g}})$ of partitions of $N_{\mathfrak{g}}$; $N_{\mathfrak{g}}$ being the dimension of the standard representation of $\mathfrak{g}$. Moreover, the partial ordering of orbits is identical to the partial ordering of the corresponding partitions: that is to say, if $\mathbf{p}_{\mathcal{O}}$ denotes the partition corresponding to a nilpotent orbit $\mathcal{O}$, then
$$\mathcal{O}' \subseteq \overline{\mathcal{O}} \quad \Longleftrightarrow \quad \mathbf{p}_{\mathcal{O}'} \leq \mathbf{p}_{\mathcal{O}}$$
where $\leq$ denotes the usual *dominance* partial ordering of partitions:
$$\mathbf{p} \leq \mathbf{q} \quad \Longleftrightarrow \quad \sum_{j=1}^{i} p_i \leq \sum_{j=1}^{i} q_i \qquad ; \quad i = 1, \dots, \min\left(length\left(\mathbf{p}\right), length\left(\mathbf{q}\right)\right) \quad .$$
So once we identify the partitions corresponding to the Richardson orbits $R_\Gamma$, $\Gamma \in \Upsilon$, we can readily identify the minimal Richardson orbits that contain a given nilpotent orbit $\mathcal{O}_{\mathbf{p}}$, $\mathbf{p} \in \mathcal{P}_\varepsilon(N_{\mathfrak{g}})$. In particular, we can do this for a special orbit $\mathcal{O}$ and its Spaltenstein dual $d(\mathcal{O})$ and thereby compute the tau signature of $\mathcal{O}$.

### 5.1. Standard Gammas.

Recall that a collection of standard Gammas is a set $\Psi$ of subsets of $\Pi$ such that the correspondence
$$\Psi \ni \Gamma \longleftrightarrow G \cdot \mathfrak{l}_\Gamma$$
is a bijection. It turns out that, almost always, if $\mathfrak{l}_\Gamma \approx \mathfrak{l}_{\Gamma'}$ as Lie algebras then $\mathfrak{l}_\Gamma$ is $G$-conjugate to $\mathfrak{l}_{\Gamma'}$. So, usually, forming a collection of standard Gammas amounts to selecting a particular $\Gamma$ out of each equivalence class
$$\Psi_{\mathfrak{l}} \equiv \{\Gamma \in \Psi \mid \mathfrak{l}_\Gamma \approx \mathfrak{l}\}$$
and a convenient way to choose as a representative of $\Psi_{\mathfrak{l}}$, the $\Gamma$ appears in the natural lexicographic ordering of $\Psi_{\mathfrak{l}}$. (For classical groups, this has the effect of selecting the $\Gamma \in \Psi_{\mathfrak{l}}$ which is maximally contінguous toward its beginning.) In the case when we have non-conjugate but isomorphic Levis, we also use lexicographic ordering to select a representative $\Gamma$ for each conjugacy class.

5.2. **Standard Gammas to partitions.** As remarked above, there is a standard parameterization of nilpotent orbits of the classical simple complex Lie algebras in terms of certain families of partitions. Below we shall describe these parameterizations explicitly on a case-by-case basis. But first I will describe a how one computes the partition corresponding to the Richardson orbit $R_\Gamma$ induced from a (parabolic with a) Levi subalgebra $\mathfrak{l}_\Gamma$, $\Gamma \in \Upsilon$. This method was gleamed from the discussion in §7.3 of Collingwood-McGovern.

Suppose first that $\mathfrak{l}_\Gamma$ is a maximal Levi subalgebra of $\mathfrak{g}$ (the general case will follow from this special case and "induction in stages"). Such a Levi will have only one simple root absent in its standard Gamma $\Gamma$, say that root is $i$, $i \in \{1, \ldots, n\}$and so will split

$$\mathfrak{l}_\Gamma = \mathfrak{gl}_i \oplus \mathfrak{g}'_{n-i}$$

where $\mathfrak{g}'_{n-i}$ is of the same Cartan type as $\mathfrak{g}$ but of rank $n-i$. In what follows we shall denote by $N_\mathfrak{g}$ the dimension of the standard representation of $\mathfrak{g}$. So

$$N_\mathfrak{g} = \begin{cases} n+1 & \text{when } \mathfrak{g} \approx A_n \\ 2n+1 & \text{when } \mathfrak{g} \approx B_n \\ 2n & \text{when } \mathfrak{g} \approx C_n \\ 2n & \text{when } \mathfrak{g} \approx D_n \end{cases}$$

Of course, the nilpotent orbits of $\mathfrak{g}$, for classical $\mathfrak{g}$ are parameterized in terms of partitions of $N_\mathfrak{g}$.

Here is how one finds the partition of $N_\mathfrak{g}$ corresponding to the nilpotent orbit induced from a nilpotent orbit $\mathcal{O}_\mathfrak{l}$ in maximal Levi subalgebra $\mathfrak{l}$ . We'll do the case where $\mathfrak{g} \not\approx A_n$ first. Write

$$\mathcal{O}_\mathfrak{l} = \mathcal{O}_{\mathfrak{gl}_i} \oplus \mathcal{O}_{\mathfrak{g}'_{n-i}}$$

and let $\mathbf{d}$ be the partition of $N_{\mathfrak{gl}_i} = i$ prescribing the orbit $\mathcal{O}_{\mathfrak{gl}_i}$ and let $\mathbf{f}$ be the partition of $N_{\mathfrak{g}'_{n-i}}$ prescribing the orbit $\mathcal{O}_{\mathfrak{g}'_{n-i}}$. If necessary we extend the partitions $\mathbf{f}$ or $\mathbf{d}$ with 0's so that they have the same number of parts, say so that *length* $(\mathbf{d}) =$ *length* $(\mathbf{f}) = m$. Next we define a new partition $\widetilde{\mathbf{p}} = [\widetilde{p}_1, \ldots, \widetilde{p}_m]$ of $N_\mathfrak{g}$ by setting

$$\widetilde{p}_i = 2d_i + f_i \quad , \quad i = 1, \ldots, m$$

With this setup in mind, we have

**Theorem 5.1** (Theorem 7.3.3 in C-M)**.** *Let $\mathfrak{g}$ be a classical simple Lie algebra of type $B$, $C$, or $D$, let $\mathfrak{l}$ be a maximal Levi subalgebra of $\mathfrak{g}$, let $\mathcal{O}_\mathfrak{l}$ be a nilpotent orbit in $\mathfrak{l}$, and let $\widetilde{\mathbf{p}}$ be the partition of $N_\mathfrak{g}$ attached to $\mathcal{O}_\mathfrak{l}$ be the construction above. Then the partition $\mathbf{p}$ of $N_\mathfrak{g}$ corresponding to $O = Ind_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$ is the $X$-collapse of $\widetilde{\mathbf{p}}$, where $X = B, C,$ or $D$ according to the Cartan type of $\mathfrak{g}$.*

*When $\mathfrak{g}$ is of type $D_n$ and the orbit $\mathcal{O}_{\mathfrak{g}'_{n-i}}$* is very even *and of numeral type $\left\{ \begin{array}{c} I \\ II \end{array} \right\}$, then $\mathcal{O} = Ind_\mathfrak{l}^\mathfrak{g}(\mathcal{O}_\mathfrak{l})$ is also very even and its type is determined by the following rules:*

- *If $i \neq n$, then numeral type of $\mathcal{O}$ is the same as that of $\mathcal{O}_{\mathfrak{g}'_{n-i}}$.*
- *If $r = 0$, then numeral of $\mathfrak{l}$ is the same as that of $\mathfrak{l}$ but differs from it if $n$ is odd. (Here we say that the Levi subalgebra $l_\Gamma \approx \mathfrak{gl}_{n-1}$ of rank $n-1$ is of numeral type $I$ if $\Gamma \cap \{n-1, n\} = \{n-1\}$ and is of numeral type $II$ if $\Gamma \cap \{n-1, n\} = \{n\}$.)*

Now consider a maximal Levi subalgebra $\mathfrak{l}_\Gamma$. Since $\mathfrak{l}_\Gamma$ is maximal, $\Gamma$ is of the form

$$\Gamma_i = \{1, 2, \ldots, n\} - \{i\} = \{1, \ldots, i-1\} \cup \{i+1, \ldots, n\}$$

We refer to $\{1, \ldots, i-1\}$ as the $A$-head of $\Gamma$ and $\{i+1, \ldots, n\}$ as the $G$-tail of $\Gamma$. Now corresponding to the decomposition

$$\mathfrak{l}_{\Gamma_i} = \mathfrak{gl}_i \oplus \mathfrak{g}'_{n-i}$$

we can regard the 0-orbit of $\mathfrak{l}_{\Gamma_i}$ as

$$\mathbf{0}_{\mathfrak{l}_{\Gamma_i}} = \mathbf{0}_{\mathfrak{gl}_i} \oplus \mathbf{0}_{\mathfrak{g}'_{n-i}}$$

The partitions of $i = N_{\mathfrak{gl}_i}$ and $N_{\mathfrak{g}'_{n-i}}$ corresponding respectively to $\mathbf{0}_{\mathfrak{gl}_i}$ and $\mathbf{0}_{\mathfrak{g}'_{n-i}}$ are, respectively

$$\mathbf{d} = (1)^i \quad , \text{ and}$$
$$\mathbf{f} = (1)^{N_{\mathfrak{g}'_r}}$$

It now quite easy to construct the partition $\widetilde{\mathbf{p}}$ and its $X$-collapse $\mathbf{p}$, which will be the partition corresponding to the Richardson orbit $Ind_{\mathfrak{l}_{\Gamma_i}}^{\mathfrak{g}}\left(\mathbf{0}_{\mathfrak{l}_{\Gamma_i}}\right)$.

What about the partitions corresponding to the Richardson orbits arising from non-maximal Levi subalgebra $\mathfrak{l}_\Gamma$? These are easily handled as well. For a standard Gamma $\Gamma \in \Upsilon$ we define its *G-tail* as the last contiguous subsequence in $\Gamma$ that ends in $n = rnk(\mathfrak{g})$. (And here we are thinking of the integers that appear in $\Gamma$ as being listed in increasing order.) If $\Gamma$ does not contain $n$ then we regard its $G$-tail as empty. The *A-head* of $\Gamma$ will be defined as the complement of the $G$-tail of $\Gamma$ in $\Gamma$. So for example, if $\Gamma = \{1, 2, 3, 5, 6, 9, 10\}$ its $G$-tail will be $\{9, 10\}$ and its $A$-head will be $\{1, 2, 3, 5, 6\}$. (We remark that if this $\Gamma$ arose in the situation where $\mathfrak{g} = B_{10}$, then the semisimple part of the corresponding Levi subalgebra would be isomorphic to $A_3 + A_2 + B_2$.)

Here's how we get the partition of $N_{\mathfrak{g}}$ corresponding to the Richardson orbit corresponding to a general $\Gamma \in \Upsilon$. The first step is to split $\Gamma$ into its $A$-head $\Gamma_A$ and its $G$-tail $\Gamma_G$. Suppose the cardinality of $\Gamma_G$ is $r$, this means the Levi subalgebra $\mathfrak{l}_\Gamma$ of $\mathfrak{g}$ has a factor $\mathfrak{g}'_r$ (i.e. a factor of the same Cartan type as $\mathfrak{g}$ but of rank $r$). The $A$-head, on the other hand, indicates that $\mathfrak{l}_\Gamma$ contains a certain standard Levi subalgebra $\mathfrak{l}_A$ of $\mathfrak{gl}_{n-r}$; the standard Levi whose semisimple part is the subalgebra of $\mathfrak{sl}_{n-r}$ generated by the simple roots indexed in $\Gamma_A$. Here's what we do: instead of trying to induce directly from $\mathbf{0}_{\mathfrak{l}_\Gamma}$ up to $\mathfrak{g}$, we first induce from $\mathfrak{l} = \mathfrak{l}_A \oplus \mathfrak{g}'_r$ up to $\mathfrak{gl}_{n-r} \oplus \mathfrak{g}'_r$. This first induction effectively ignores the $\mathfrak{g}'_r$ factor and replaces the $\mathbf{0}$-orbit of $\mathfrak{l}_A$ with the corresponding Richardson orbit in $\mathfrak{gl}_{n-r}$. Suppose we have we figured out how to attach a partition $\mathbf{d}_A$ of $n-r$ to a standard Gamma $\Gamma_A$ for $\mathfrak{sl}_{\,n-r}$ in such a way that $\mathbf{d}_A$ corresponds to the Richardson orbit of the standard Levi subalgebra $\mathfrak{l}_{\Gamma_A}$ of $\mathfrak{sl}_{n-r}$, thne we can readily apply Collingwood-McGovern's Theorem 7.3.3 (our Theorem 3.1) to get the right partition for the Richardson orbit of $\mathfrak{l}_\Gamma$ in $\mathfrak{g}$. A little more explicitly, once we induce up to the trivial orbit of $\mathfrak{l}_\Gamma$ up to $\mathfrak{gl}_{n-r} \oplus \mathfrak{g}'_r$, the next stage, induction from the maximal Levi subalgebra $\mathfrak{gl}_{n-r} \oplus \mathfrak{g}'_r$ to $\mathfrak{g}$, can be carried out using the partition corresponding to $Ind_{\mathfrak{l}_A}^{\mathfrak{gl}_{n-r}}(\mathbf{0}_{\mathfrak{l}_A})$ as $\mathbf{d}$ and the partition $(1)^{N_{\mathfrak{g}'_r}}$ corresponding to the trivial representation of $\mathfrak{g}'_r$ as $\mathbf{f}$ and applying Theorem 3.1.

It just remains to clarify how we go from $\Gamma_A$ to the partition $\mathbf{d}$ of $n-r$. This can be accomplished by applying a Theorem of Kraft, Ozeki and Wakimoto (Theorem 7.2.3 in [C-M]). We'll simply re-state that result in terms of the standard Gammas $\Gamma_A$

**Theorem 5.2.** *Let $\Gamma_A$ be a standard Gamma for $\mathfrak{gl}_{n-r}$. Let $(i_1, \ldots, i_k)$ be the lengths of the contiguous subsequences of $\Gamma$. Set*

$$\mathbf{q}_{\Gamma_A} = [i_1 + 1, i_2 + 1, \ldots, i_k + 1, 1, \ldots, 1]$$

*where we add as many 1's after the $i_k + 1$ until we end up with a partition of $n-r$. Then the transpose $\mathbf{q}^t_{\Gamma_A}$ of $\mathbf{q}_{\Gamma_A}$ will be the partition of $n-r$ corresponding to the Richardson orbit $Ind_{\mathfrak{l}_\Gamma}^{\mathfrak{sl}_{n-r}}(\mathbf{0}_{\mathfrak{l}_\Gamma})$.*

*Remark* 5.3. The discussion above also tells us how to figure out the Richardson orbits $Ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}$ when $\mathfrak{g}$ is of type $A$. However, the algorithm David gives on the $A_n$ subpage of [NOLSID] already accomplishes this quite succinctly (and of course is entirely equivalent).

*Remark* 5.4. We should note here two conventions that come into play when applying Theorem 3.1 in the extreme cases when (i) $rank(\mathfrak{g}'_r) = rank(\mathfrak{g}) - 1$ and (ii) $rank(\mathfrak{g}'_r) = 0$ and $\mathfrak{g}$ is of type $B$. In situation (i), the convention is to take the partition corresponding to the trivial orbit of $\mathfrak{l}_A \approx \mathfrak{gl}_1$ to be [1] before applying Theorem 3.1 In situation (ii), the convention is takes the trivial orbit of $\mathfrak{g}'_r \approx B_0$ to be [1] before applying 7.3.3.

5.3. **Computing the tau signatures of special orbits.** Up to this point we haven't had to say exactly which partitions of $N_{\mathfrak{g}}$ are parameterize the nilpotent orbits of $\mathfrak{g}$. We'll do that now case-by-case, and at the same time describe how we identify special orbits and their tau signatures.

5.4. $A_n$**.** The nilpotent orbits of $A_n \approx \mathfrak{sl}_{n+1}$ are in a one-to-one correspondence with partitions of $n+1$. For $A_n$, every nilpotent orbit is both special and Richardson. The way to recover the $\Gamma$ for the Levi subalgebra inducing a Richardson orbit whose corresponding partition is $\mathbf{p}$ is to simply to take its transpose, subtract 1 from each entry of $\mathbf{p}^t$. Once you have $\mathbf{p}^t$ you recover the corresponding $\Gamma$ by forming a list of integers between 1 and $n$ in such a way that the first contiguous subsequence is $1, 2, \ldots, (\mathbf{p}^t)_1$, the second contiguous sequence is $(\mathbf{p}^t)_1 + 2, (\mathbf{p}^t)_1 + 3, \ldots, (\mathbf{p}^t)_1 + (\mathbf{p}^t)_2 + 1$, and so on. If we denote by $\Gamma_{\mathbf{p}}$ the standard Gamma so obtained, we can say that the correspondences

$$\mathbf{p} \longrightarrow \Gamma_{\mathbf{p}}$$
$$\mathbf{p}^t \longrightarrow \Gamma_{\mathbf{p}^t}$$

furnish us with the tau signature $(\Gamma_{\mathbf{p}}, \Gamma_{\mathbf{p}^t})$ of the special nilpotent orbit corresponding to the partition $\mathbf{p}$. For each special orbit is Richardson, so set of minimal Richardson orbits containing it is the orbit itself, and similarly for its Spaltenstein dual (which will be the nilpotent orbit corresponding to the partition $\mathbf{p}^t$).

5.5. $B_n$**.** The nilpotent orbits of $B_n \approx SO(2n+1)$ are in one-to-one correspondence with partitions of $2n+1$ such that even parts, if they occur, occur with even multiplicity. [C-M] denotes this set of partitions by $\mathcal{P}_1(2n+1)$. There are several ways of characterizing the special nilpotent orbits of $B_n$. The one I adopted in my computations was the following. Define the Spaltenstein duality map $d$ as sending a partition $\mathbf{p}$ in $\mathcal{P}_1(2n+1)$ to its transpose $\mathbf{p}^t$ and then taking its $B$-collapse, which will be the largest partition in $\mathcal{P}_1(2n+1)$ that is dominated by $\mathbf{p}^t$. The $B$-collapse of $\mathbf{p}^t$ (or any partition of $2n+1$) can be computed directly from $\mathbf{p}^t$ using Lemma 6.3.3 in [C-M]. The special nilpotent orbits for $B_n$ are can then be identified as the partitions $\mathbf{p} \in \mathcal{P}_1(2n+1)$ such that $d \circ d(\mathbf{p}) = \mathbf{p}$.

Since we already have a means of figuring out the partition $\mathbf{p}_\Gamma$ in $P_1(2n+1)$ that corresponds to a Richardson orbit

$$R_\Gamma = Ind_{\mathfrak{l}_\Gamma}^{\mathfrak{g}}(\mathbf{0}_{\mathfrak{l}_\Gamma})$$

and because the closure relations of the nilpotent orbits is consistent with the dominance ordering of the corresponding partitions, we can use the dominance ordering of partitions to figure out exactly which $\Gamma$'s lead to a Richardson orbit $R_\Gamma$ whose closure contains a given special nilpotent orbit;

$$\mathbf{p}_{\mathcal{O}} < \mathbf{p}_\Gamma \Longleftrightarrow \mathcal{O} \subset \overline{R_\Gamma}$$

And the dominance ordering of partitions will also identify which of these Richardson orbits such that $\mathcal{O}_{\mathbf{p}} \subset \overline{R_\Gamma}$ are minimal (i.e., not containing any other with the same property). In this way, we can readily determine the minimal $\Gamma \in \Upsilon$ such that $\mathcal{O} \subset \overline{R_\Gamma}$ and get the first half of the tau signature of $\mathcal{O}$. Doing the same thing for the Spaltenstein dual $d(\mathcal{O}) = \mathcal{O}_{d(\mathbf{p}_{\mathcal{O}})}$ of $\mathcal{O}$ yields the second half of the tau signature of $\mathcal{O}$.

5.6. $C_n$**.** The nilpotent orbits of $C_n \approx Sp(2n)$ are in one-to-one correspondence with the partitions of $2n$ such that the odd parts, if they occur, occur with even multiplicity. Collingwood and McGovern denote this set of partitions by $\mathcal{P}_{-1}(2n)$. The identification of special nilpotent orbits and their tau signatures is carried out in the same manner as in the $B_n$ case (just utilizing $C$-collapses in place of $B$-collapses).

5.7. $D_n$**.** The nilpotent orbits of $D_n \approx SO(2n)$ are in a nearly one-to-one correspondence with partitions of $2n$ for which the even parts occur with even multiplicity. Collingwood-McGovern denotes this set of partitions is denoted by $\mathcal{P}_1(2n)$. The hedge "nearly one-to-one" is due to the fact that to a *very even* partition, which is a partition $\mathbf{p} \in \mathcal{P}_1(2n)$ consisting of **only** even parts, there corresponds two distinct nilpotent potent orbits, usually denoted by $\mathcal{O}_{\mathbf{p}}^I$ and $\mathcal{O}_{\mathbf{p}}^{II}$ or some such notation involving the *numerals I* and $II$. But so long as a partition $\mathbf{p} \in \mathcal{P}_1(2n)$ is not very even, the correspondence between partitions and nilpotent orbits is one-to-one.

Luckily the complications arising from the even-even orbits are relatively mild. In particular, the partial ordering of the partitions in $\mathcal{P}_1(2n)$ still gives the correct closure relations for the corresponding orbit, one just has to understand that if $\mathbf{p}$ is a very even partition then the orbits $\mathcal{O}_{\mathbf{p}}^I$ and $\mathcal{O}_{\mathbf{p}}^{II}$ must be regarded as incomparable (which is in fact true with respect to the closure ordering of the nilpotent orbits).

Now it turns out that there are only a few cases where a Richardson orbit $R_\Gamma$ turns out to be very even: for example, it does happen for the Levis

$$(A_{n-1})' \longleftrightarrow \Gamma = (1, \ldots, n-1) \qquad \text{and} \qquad (A_{n-1})'' \longleftrightarrow \Gamma = (1, \ldots, n-2, n)$$

when n is even. More generally, this happens whenever the G-tail of a standard $\Gamma$ is trivial and the partition corresponding to the A-head has only even multiplicities. At any rate, dealing with this complication just amounted to figuring out when and how to distinguish between very even Richardson orbit pairs. It actually was not hard to kill both birds with one stone; we simply adopted a computationally innocuous convention in which the numeral I very even orbits are represented by truncated very even partitions (i.e., with only non-zero parts included) and the numeral II very even orbits are represented by very even partitions with one trailing 0. Thus , for example, the Richardson orbit corresponding to the $(A_{n-1})'$ Levi was distinguished from that of the $(A_{n-1})''$ Levi as follows

$$\mathcal{O}^{I}_{[(2)^n]} \longleftrightarrow [(2)^n]$$
$$\mathcal{O}^{II}_{[(2)^n]} \longleftrightarrow [(2)^n\,, 0]$$

I remark that this hack was harmless to the partial ordering computations, yet turned out also to be very useful in getting the right Spaltenstein duality map. For with this convention it was very easy to implement the convention of [ CM] (in which one changes the numeral type of a very even orbit upon taking its Spaltenstein dual whenever $\mathfrak{g} \approx D_n$ and $n$ is odd).

After dealing with the very even orbits in the manner described above, the computation of the tau signatures of the special nilpotent orbits in $D_n$ was carried out in the same fashion as for $B_n$ and $C_n$.

## 6. Tables

Below I give the orbit - cell correspondence for the big block of $D_8$. Note the use of the convention of adding a trailing zero to distinguish type II from type I orbits (for very even partitions).

```
[15,1] <----> cell #'s 0,1,2,4
[13,3] <----> cell #'s 3,5,6,10
[13,1,1,1] <----> cell #'s 8,18
[11,5] <----> cell #'s 7,11,13,15
[11,3,1,1] <----> cell #'s 9,12,19,20,21,38
[11,1,1,1,1,1] <----> cell #'s 67,72
[9,7] <----> cell #'s 14,16,24,28
[9,5,1,1] <----> cell #'s 17,22,33,39
[9,3,3,1] <----> cell #'s 23,32,35,41,43,46,48,65
[9,3,2,2] <----> cell #'s 37,50,51,53
[9,3,1,1,1,1] <----> cell #'s 68,73,74,114
[9,1,1,1,1,1,1,1] <----> cell #'s 102,131
[8,8] <----> cell #'s 26,30
[8,8,0] <----> cell #'s 25,29
[7,7,1,1] <----> cell #'s 27,31,34,40,64
[7,5,3,1] <----> cell #'s 36,42,44,47,49,56,66,71,77
[7,5,2,2] <----> cell #'s 52,54,57,79
[7,5,1,1,1,1] <----> cell #'s 69,75,81,115
[7,3,3,3] <----> cell #'s 55,80
[7,3,3,1,1,1] <----> cell #'s 70,76,82,83,90,92,95,116,117,118,120,123
[7,3,2,2,1,1] <----> cell #'s 84,100,119,124,143,145
[7,3,1,1,1,1,1,1] <----> cell #'s 103,132,144,148
[7,1,1,1,1,1,1,1,1,1] <----> cell #'s 161,169
[6,6,3,1] <----> cell #'s 45,58,60,78
```

```
[6,6,2,2] <----> cell #'s 61,87
[6,6,2,2,0] <----> cell #'s 59,86
[6,6,1,1,1,1] <----> cell # 89
[5,5,5,1] <----> cell #'s 62,85
[5,5,3,3] <----> cell #'s 63,88,97,98,99
[5,5,3,1,1,1] <----> cell #'s 91,93,96,104,121,126,133
[5,5,2,2,1,1] <----> cell #'s 105,134,146
[5,5,1,1,1,1,1,1] <----> cell #'s 107,136,149
[5,3,3,3,1,1] <----> cell #'s 94,101,106,122,125,135,147,151,156
[5,3,3,1,1,1,1,1] <----> cell #'s 108,112,137,139,150,152,153,158,164
[5,3,2,2,2,2] <----> cell #'s 172,173,174,175
[5,3,2,2,1,1,1,1] <----> cell #'s 159,167,181,182
[5,3,1,1,1,1,1,1,1,1] <----> cell #'s 162,170,185,186
[5,1,1,1,1,1,1,1,1,1,1,1] <----> cell #'s 197,200
[4,4,4,4] <----> cell #'s 109,127
[4,4,4,4,0] <----> cell #'s 110,128
[4,4,3,3,1,1] <----> cell #'s 111,129,138,141,154
[4,4,3,1,1,1,1,1] <----> cell #'s 155,165
[4,4,2,2,2,2] <----> cell #'s 178,179
[4,4,2,2,2,2,0] <----> cell #'s 176,177
[4,4,2,2,1,1,1,1] <----> cell # 183
[4,4,1,1,1,1,1,1,1,1] <----> cell #'s 192
[3,3,3,3,3,1] <----> cell #'s 157,166
[3,3,3,3,2,2] <----> cell # 180
[3,3,3,3,1,1,1,1] <----> cell #'s 113,130,140,142,160,168,184,187
[3,3,3,1,1,1,1,1,1,1] <----> cell #'s 163,171,193,194,195
[3,3,2,2,2,2,1,1] <----> cell #'s 188,189,190,191,202
[3,3,2,2,1,1,1,1,1,1] <----> cell #'s 196,199,207
[3,3,1,1,1,1,1,1,1,1,1,1] <----> cell #'s 198,201,208
[3,1,1,1,1,1,1,1,1,1,1,1,1,1] <----> cell #'s 210,211
[2,2,2,2,2,2,2,2] <----> cell #'s 205,206
[2,2,2,2,2,2,2,2,0] <----> cell #'s 203,204
[2,2,2,2,2,2,1,1,1,1] <----> cell # 209
[2,2,2,2,1,1,1,1,1,1,1,1] <----> cell # 212
[2,2,1,1,1,1,1,1,1,1,1,1,1,1] <----> cell # 213
[1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1] <----> cell # 214
```

| B-C Type | Cell #'s | GraphIsoClass |
|---|---|---|
| E8 | 0 | 1* |
| E8(a1) | 1 | 8* |
| E8(a2) | 2 | 35-A |
| E8(a3) | 3 | 196-A |
| | 5 | 196-A |
| E8(a4) | 4 | 260-A |
| | 7 | 260-A |
| E8(b4) | 6 | 560-A |
| | 8 | 560-A |
| | 9 | 560-A |
| E8(a5) | 10 | 1100-A |
| | 13 | 1100-A |
| | 17 | 1100-A |
| E7(a1) | 11 | 567-A |
| E8(b5) | 12 | 3192-A |
| | 14 3752-A | |
| E8(a6) | 15 | 4025-A |
| | 18 | 2625* |
| D7(a1) | 16 | 3240-A |
| | 19 | 3240-A |
| | 20 | 3240-A |
| | 21 | 3240-A |
| | 22 | 3240-A |
| E8(b6) | 23 | 3640-A |
| | 25 | 3640-A |
| | 27 | 3640-A |
| E7(a3) | 24 | 3240-B |
| | 29 | 3240-B |
| E6(a1)+A1 | 26 | 8192-A |
| | 30 | 8192-A |
| D7(a2) | 28 | 7560-A |
| | 31 | 5040-A |
| | 35 | 7560-A |
| E6 | 33 | 525-A |
| D5+A2 | 32 | 4536-A |
| | 38 | 4536-A |
| | 39 | 4536-A |
| | 45 | 4536-A |
| E6(a1) | 34 | 3500-A |
| | 37 | 3500-A |
| E7(a4) | 36 | 6075-A |
| | 41 | 6075-A |
| | 43 | 6075-A |
| A6+A1 | 40 | 2835-A |
| D6(a1) | 44 | 8800-A |
| | 49 | 8800-A |
| A6 | 42 | 4200-A |
| | 46 | 4200-A |
| | 48 | 4200-A |
| | 50 | 4200-A |
| E8(a7) | 47 | 22778 |
| | 51 | 38766 |
| | 53 | 46676 |

| | | |
|---|---|---|
| D5 | 54 | 2100* |
| E6(a3) | 55 | 8800-a |
| | 63 | 8800-a |
| D4+A2 | 52 | 4200-a |
| | 57 | 4200-a |
| | 59 | 4200-a |
| | 62 | 4200-a |
| A4+A2+A1 | 60 | 2835-a |
| D5(a1)+A1 | 58 | 6075-a |
| | 64 | 6075-a |
| | 65 | 6075-a |
| A4+A2 | 56 | 4536-a |
| | 66 | 4536-a |
| | 68 | 4536-a |
| | 69 | 4536-a |
| A4+2A1 | 61 | 7560-a |
| | 67 | 7560-a |
| | 70 | 5040-a |
| D5(a1) | 71 | 3500-a |
| | 79 | 3500-a |
| A4+A1 | 72 | 8192-a |
| | 80 | 8192-a |
| D4(a1)+A2 | 73 | 3640-a |
| | 78 | 3640-a |
| | 81 | 3640-a |
| A4 | 74 | 3240-b |
| | 82 | 3240-b |
| A3+A2 | 75 | 3240-a |
| | 76 | 3240-a |
| | 83 | 3240-a |
| | 84 | 3240-a |
| | 86 | 3240-a |
| D4(a1)+A1 | 87 | 4025-a |
| | 89 | 2625* |
| D4 | 85 | 525-a |
| D4(a1) | 88 | 3752-a |
| | 90 | 3192-a |
| 2A2 | 77 | 1100-a |
| | 91 | 1100-a |
| | 92 | 1100-a |
| A3 | 94 | 567-a |
| A2+2A1 | 93 | 560-a |
| | 95 | 560-a |
| | 96 | 560-a |
| A2+A1 | 97 | 260-a |
| | 99 | 260-a |
| A2 | 98 | 196-a |
| | 100 | 196-a |
| 2A1 | 101 | 35-a |
| A1 | 102 | 8 |
| 0 | 103 | 1 |

## References


[ BV] D. Barbasch and D. Vogan, *Unipotent representations of complex semisimple groups*, Ann. of Math. **121** (1985), 41-110.

[ CM] D. Collingwood and W. McGovern, *Nilpotent Orbits in Semisimple Lie Algebras*, Van Nostrand Reinhold, New York (1993).

[ Sp1] N. Spaltenstein, *A property of special representations of Weyl groups*, J. Reine Angew. Math. **343** (1983), 212-220.

[ Sp2] N. Spaltenstein, *Classes unipotentes et sous-groupes de Borel*, Lec. Notes Math. **946**, Springer-Verlag, New York, 1982.

[ Vo] D. Vogan (Atlas Wiki entry, August 2007).